

Density Matrix Equations and Atomic Spectroscopy

2022 Winter School on Atomic and Molecular Physics

KAIST, Feb. 10, 2022

Heung-Ryoul Noh

Department of Physics, Chonnam National University,

Gwangju 61186, Korea

Overview

*How to calculate **optical response** in
atomic and laser spectroscopy using
density matrix equations?*

- Transmission
- Polarization rotation
- Four-wave mixing

- Method: Analytically or Numerically
- Transitions: Closed or Open
- Polarization

Light

- classical
- not strong
- 1 or 2 frequencies

Atoms

- dilute vapor cell or cold atoms
- 2, 3, or multilevel

*1. What is susceptibility, absorption,
dispersion, and cross section ?*

*2. How to calculate internal
states of atoms ?*

- Solving density matrix equations

3. How to calculate spectra ?

- Open system (temporal or transit relaxation)
- Doppler averaging

References

1. C. Cohen-Tannoudji, et al.
“Atom-Photon Interactions”
2. C. Cohen-Tannoudji, et al.
“Photons and Atoms”
3. D. Meschede, “Optics, Light and Lasers”
4. C. J. Foot, “Atomic Physics”
5. P. R. Berman and V. S. Malinovsky
“Principles of Laser Spectroscopy and Quantum Optics”
6. Daniel A. Steck, “Rubidium 87 D Line Data”
<http://steck.us/alkalidata>
7. Daniel A. Steck, “Quantum and Atom Optics”
<http://steck.us/alkalidata>
8. B. R. Mollow, “Stimulated Emission and Absorption near Resonance for Driven System”, Phys. Rev. A 5, 2217 (1972).

Contents

- **Two-Level Atoms**

- One field (Optical Bloch equations)
- One field (Linear absorption)
- One field (Analytical solutions to Optical Bloch equations)
- Two fields (Susceptibility)
- Two fields (Polarization spectroscopy for the $F_g=0 \rightarrow F_e=1$ transition)

- **Multi-Level Atoms**

- Density matrix equations
- Electromagnetically induced transparency and absorption
- Wave mixing and rate equations
- Elliptic polarization and arbitrary quantization axis

Contents

- **Two-Level Atoms**

- **One field (Optical Bloch equations)**

- One field (Linear absorption)

- One field (Analytical solutions to Optical Bloch equations)

- Two fields (Susceptibility)

- Two fields (Polarization spectroscopy for the $F_g=0 \rightarrow F_e=1$ transition)

- **Multi-Level Atoms**

- Density matrix equations

- Electromagnetically induced transparency and absorption

- Wave mixing and rate equations

- Elliptic polarization and arbitrary quantization axis

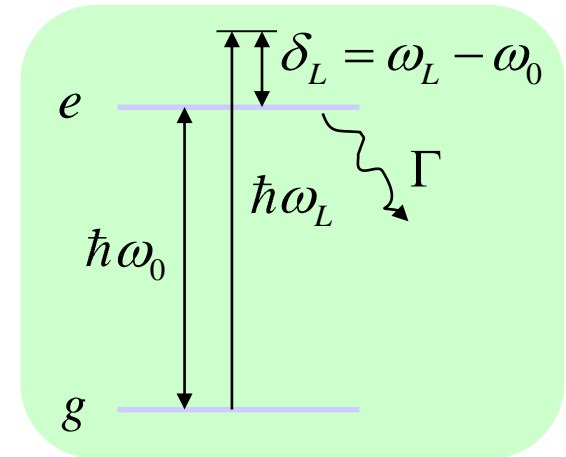
Two Level Atoms

Two-level atoms

- closed transition
- one laser frequency

Density matrix equation

$$i\hbar\dot{\rho} = [H_A - \vec{d} \cdot \vec{E}_0 \cos \omega_L t, \rho] + i\hbar(\dot{\rho})_{\text{sp}}$$



$$\dot{\rho}_{ee} = i\Omega_1 \cos \omega_L t (\rho_{eg} - \rho_{ge}) - \Gamma \rho_{ee}$$

$$\dot{\rho}_{gg} = -i\Omega_1 \cos \omega_L t (\rho_{eg} - \rho_{ge}) + \Gamma \rho_{ee}$$

$$\dot{\rho}_{ge} = i\omega_0 \rho_{ge} - i\Omega_1 \cos \omega_L t (\rho_{ee} - \rho_{gg}) - \frac{\Gamma}{2} \rho_{ge}$$

$$\dot{\rho}_{eg} = -i\omega_0 \rho_{eg} + i\Omega_1 \cos \omega_L t (\rho_{ee} - \rho_{gg}) - \frac{\Gamma}{2} \rho_{eg}$$

$$\Omega_1 = -\frac{\vec{d}_{ge} \cdot \vec{E}_0}{\hbar}$$

Rabi Frequency

$$\vec{d}_{ge} = \langle g | \vec{d} | e \rangle = \langle e | \vec{d} | g \rangle$$

When elastic collisions dephase the atomic dipole

$$\left(\frac{d\sigma_{ge}}{dt} \right)_{\text{coll}} = -\gamma \sigma_{ge} \Rightarrow \dot{\rho}_{ge} = i\omega_0 \rho_{ge} - i\Omega_1 \cos \omega_L t (\rho_{ee} - \rho_{gg}) - \gamma' \rho_{ge} \quad \gamma' = \frac{\Gamma}{2} + \gamma$$

In a dilute atomic vapor $\gamma = 0$ and $\gamma' = \frac{\Gamma}{2}$

Relaxation Times and Rotating Wave Approx.

Longitudinal relaxation time $T_1 = \frac{1}{\Gamma}$

Transverse relaxation time $T_2 = \frac{1}{\gamma'}$ $T_2 = \frac{2}{\Gamma}$ (dilute atomic vapor)

$$\Gamma = \frac{1}{3\pi\epsilon_0} \frac{\omega_0^2}{\hbar c^3} d_{ge}^2$$

C. Cohen-Tannoudji,
“Atom-Photon Interactions”
p 515~518

Rotating wave approximation:
Neglecting non-resonant terms

$$\begin{aligned} -\vec{d} \cdot \vec{E}_0 \cos \omega_L t &= -\vec{d}_{ge} \cdot \vec{E}_0 \frac{1}{2} (|e\rangle\langle g| + |g\rangle\langle e|)(e^{i\omega_L t} + e^{-i\omega_L t}) \\ &\Rightarrow \frac{1}{2} \hbar \Omega_1 (|e\rangle\langle g| e^{-i\omega_L t} + |g\rangle\langle e| e^{i\omega_L t}) \end{aligned}$$

Optical Bloch Equations

Transformation with slowly-varying variables

$$\rho_{eg} = \sigma_{eg} e^{-i\omega_L t}$$

$$\rho_{ge} = \sigma_{ge} e^{i\omega_L t}$$

$$\rho_{ee} = \sigma_{ee}, \rho_{gg} = \sigma_{gg}$$

$$u = \frac{1}{2}(\sigma_{ge} + \sigma_{eg})$$

$$v = \frac{1}{2i}(\sigma_{ge} - \sigma_{eg})$$

$$w = \frac{1}{2}(\sigma_{ee} - \sigma_{gg})$$

$$\sigma_{ge} = u + iv$$

$$\sigma_{eg} = u - iv$$

(u, v, w) Bloch vector

$$\sigma_{gg} + \sigma_{ee} = 1$$

$$\dot{\sigma}_{ee} = i \frac{\Omega_1}{2} (\sigma_{eg} - \sigma_{ge}) - \Gamma \sigma_{ee}$$

$$\dot{\sigma}_{gg} = -i \frac{\Omega_1}{2} (\sigma_{eg} - \sigma_{ge}) + \Gamma \sigma_{ee}$$

$$\dot{\sigma}_{ge} = -i\delta_L \sigma_{ge} - i \frac{\Omega_1}{2} (\sigma_{ee} - \sigma_{gg}) - \frac{\Gamma}{2} \sigma_{ge}$$

$$\dot{\sigma}_{eg} = i\delta_L \sigma_{eg} + i \frac{\Omega_1}{2} (\sigma_{ee} - \sigma_{gg}) - \frac{\Gamma}{2} \sigma_{eg}$$

$$\langle \vec{d} \rangle = \text{Tr}(\rho \vec{d}) = \vec{d}_{ge} (\rho_{ge} + \rho_{eg})$$

$$= \vec{d}_{ge} (\sigma_{ge} e^{i\omega_L t} + \sigma_{eg} e^{-i\omega_L t})$$

$$= 2\vec{d}_{ge} (u \cos \omega_L t - v \sin \omega_L t)$$

Optical Bloch Equations

Optical Bloch Equations

$$\begin{aligned}\dot{u} &= \delta_L v - \frac{\Gamma}{2} u \\ \dot{v} &= -\delta_L u - \Omega_1 w - \frac{\Gamma}{2} v \\ \dot{w} &= \Omega_1 v - \Gamma w - \frac{\Gamma}{2}\end{aligned}$$

Analytic solution
(*Torrey's solution*)

$$q(t) = Ae^{-at} + \left[B \cos st + \frac{C}{s} \sin st \right] e^{-bt} + D$$

$q = u, v, w$

Steady-state values

a, b, s depend only on δ, Ω_1, Γ

H. C. Torrey, Phys. Rev. 76, 1059 (1949).

Steady State Solutions

Steady-State Regime

$$u_{\text{st}} = \frac{\Omega_1}{2} \frac{\delta_L}{\delta_L^2 + (\Gamma^2/4) + (\Omega_1^2/2)}$$
$$v_{\text{st}} = \frac{\Omega_1}{2} \frac{\Gamma/2}{\delta_L^2 + (\Gamma^2/4) + (\Omega_1^2/2)}$$
$$w_{\text{st}} = -\frac{\delta_L^2 + (\Gamma^2/4)}{2[\delta_L^2 + (\Gamma^2/4) + (\Omega_1^2/2)]}$$

$$\sigma_{ee}^{\text{st}} = \frac{\Omega_1^2}{4} \frac{1}{\delta_L^2 + (\Gamma^2/4) + (\Omega_1^2/4)}$$
$$\sigma_{gg}^{\text{st}} = \frac{\delta_L^2 + (\Gamma^2/4) + (\Omega_1^2/4)}{\delta_L^2 + (\Gamma^2/4) + (\Omega_1^2/2)}$$

Saturation parameter

$$s = \frac{\Omega_1^2/2}{\delta^2 + \Gamma^2/4}$$

$$u_{\text{st}} = \frac{\delta_L}{\Omega_1} \frac{s}{1+s} \quad \sigma_{ee}^{\text{st}} = \frac{s}{2(1+s)}$$
$$v_{\text{st}} = \frac{\Gamma}{2\Omega_1} \frac{s}{1+s} \quad \sigma_{gg}^{\text{st}} = \frac{2+s}{2(1+s)}$$
$$w_{\text{st}} = -\frac{1}{2(1+s)}$$

Saturation Intensity

Saturation parameter on resonance $\delta = 0$

$$s_0 = \frac{I}{I_s} = \frac{2\Omega_1^2}{\Gamma^2} \quad \leftarrow \Omega_1 = -\frac{\vec{d}_{ge} \cdot \vec{E}_0}{\hbar}$$

$$\Rightarrow \frac{I}{I_s} = \frac{2}{\Gamma^2} \left(\frac{d_{ge} E_0}{\hbar} \right)^2 \quad \leftarrow \begin{cases} I = \frac{1}{2} \epsilon_0 c E_0^2 \\ \Gamma = \frac{1}{3\pi\epsilon_0} \frac{\omega_0^2}{\hbar c^3} d_{ge}^2 \end{cases}$$

$$\Rightarrow \frac{I}{I_s} = \frac{2}{\Gamma^2 \hbar^2} \frac{3\pi\epsilon_0 \hbar c^3 \Gamma}{\omega_0^3} \frac{2I}{\epsilon_0 c}$$

Saturation intensity

$$I_s = \frac{\pi \hbar c}{3\lambda^3} \Gamma$$

Saturation parameter

$$s = \frac{\Omega_1^2/2}{\delta^2 + \Gamma^2/4} = \frac{I/I_s}{1 + 4\delta^2/\Gamma^2}$$

Photon Absorption

Work done on atomic electron by the driving fields

$$dW = q\vec{E}_0 \cos \omega_L t \cdot d\vec{r}$$

$$\left\langle \frac{dW}{dt} \right\rangle = \vec{E}_0 \cos \omega_L t \cdot \left\langle \dot{\vec{d}} \right\rangle \quad \leftarrow \quad \left\langle \vec{d} \right\rangle = 2\vec{d}_{ge} (u \cos \omega_L t - v \sin \omega_L t)$$

Mean absorbed power

$$\begin{aligned} \left\langle \frac{dW}{dt} \right\rangle &= -2\vec{d}_{ge} \cdot \vec{E}_0 \omega_L \left[\overline{\cos^2 \omega_L t} \, v + \overline{\sin \omega_L t \cos \omega_L t} \, u \right] \\ &= \hbar \Omega_1 \omega_L v \end{aligned}$$

Mean number of photons absorbed per unit time by the atom

$$\left\langle \frac{dN}{dt} \right\rangle = \Omega_1 v = \frac{\Gamma}{2} \frac{s}{1+s}$$

Photon absorption

$$\dot{w} = \Omega_1 \nu - \Gamma w - \frac{\Gamma}{2}$$

$$w = \frac{1}{2}(\sigma_{ee} - \sigma_{gg}) = \sigma_{ee} - \frac{1}{2}$$

$$\dot{\sigma}_{ee} = \Omega_1 \nu - \Gamma \sigma_{ee} = \left\langle \frac{dN}{dt} \right\rangle - \Gamma \sigma_{ee}$$

$$\left\langle \frac{dN}{dt} \right\rangle_{\text{st}} = \Gamma \sigma_{ee}^{\text{st}} = \Omega_1 \nu = \frac{\Gamma}{2} \frac{s}{1+s}$$

Number of photons
absorbed per unit time

Number of photons emitted
spontaneously per unit time

Classical Cross Section

C. Cohen-Tannoudji, et al.
“Photons and Atoms”

Lorentz oscillator model

$$m\ddot{z} = -m\omega_0^2 z + \frac{q^2}{6\pi\epsilon_0 c^3} \ddot{z} + qE_0 \cos \omega t$$

$$z = \text{Re}(z_0 e^{-i\omega t})$$

$$z_0 = \frac{qE_0}{m} \frac{1}{\omega_0^2 - \omega^2 - i\gamma \frac{\omega^3}{\omega_0^2}}$$

Mean energy radiated per unit time

$$\frac{\overline{dW}}{dt} = \frac{1}{3} \frac{q^2}{4\pi\epsilon_0} \frac{|z_0|^2 \omega^4}{c^3} = \frac{1}{3} \frac{q^2}{4\pi\epsilon_0} \frac{q^2 E^2}{m^2} \frac{\omega^4}{c^3} \frac{1}{\left[\omega_0^2 - \omega^2\right]^2 + \gamma^2 \frac{\omega^6}{\omega_0^4}}$$

Cross section $\sigma(\omega)$

$$\sigma(\omega) \Rightarrow \frac{\text{Energy/time}}{\text{Energy/time/Area}} = \text{Area}$$

$$E(t) = E_0 \cos \omega t$$



Classical decay rate

$$\gamma = \frac{q^2}{6\pi\epsilon_0} \frac{\omega_0^2}{mc^3} = \frac{2}{3} \frac{r_0 \omega_0^2}{c} = \frac{4\pi}{3} \frac{r_0}{\lambda_0} \omega_0$$

$$r_0 = \frac{1}{4\pi\epsilon_0} \frac{q^2}{mc^2} = 2.8 \times 10^{-15} \text{ m}$$

Classical electron radius

$$a_0, \tilde{\lambda}_c, r_0$$

$$\alpha = \frac{1}{4\pi\epsilon_0} \frac{e^2}{\hbar c}$$

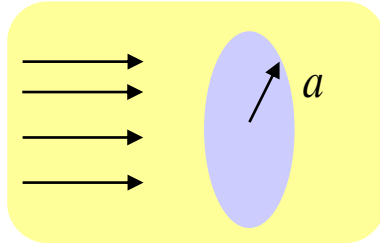
Incident flux

$$\overline{\phi}_i = \frac{1}{2} \epsilon_0 c E^2$$

$$\sigma(\omega) = \frac{1}{\overline{\phi}_i} \frac{\overline{dW}}{dt} = \frac{8\pi}{3} r_0^2 \frac{\omega^4}{\left[\omega_0^2 - \omega^2\right]^2 + \gamma^2 \frac{\omega^6}{\omega_0^4}}$$

Classical Cross Section

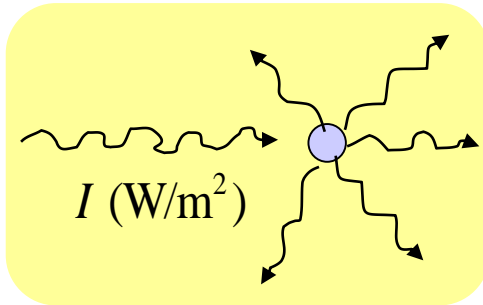
Classical particles



Scattering rate = Incident flux \times Area

Area \rightarrow Cross section $\sigma = \pi a^2$

Atoms (classical model
Near resonance)



Scattered power (W) = $I \times \sigma(\omega)$

Near resonance $\omega \cong \omega_0$

$$\omega_0^2 - \omega^2 = (\omega_0 + \omega)(\omega_0 - \omega) \cong -2\omega_0\delta, \quad \gamma \frac{\omega^6}{\omega_0^4} \cong \gamma\omega_0^2$$

$$\sigma(\omega) = \frac{8\pi}{3} r_0^2 \frac{\omega^4}{\left[\omega_0^2 - \omega^2\right]^2 + \gamma^2 \frac{\omega^6}{\omega_0^4}} \cong \frac{2\pi}{3} r_0^2 \frac{\omega_0^2}{\left(\omega - \omega_0\right)^2 + \frac{\gamma^2}{4}}$$

$$\sigma(\omega) = \frac{3\lambda^2}{2\pi} \frac{1}{1 + 4\left(\omega - \omega_0\right)^2 / \gamma^2}$$

Cross Section

Scattered photon energy per unit time
= incident irradiance \times cross section

$$\left\langle \frac{dN}{dt} \right\rangle_{\text{st}} \hbar\omega = \frac{\Gamma}{2} \frac{s}{1+s} \hbar\omega = \frac{\Gamma}{2} \frac{I/I_s}{1+I/I_s + 4\delta_L^2/\Gamma^2} \hbar\omega \Rightarrow I\sigma$$

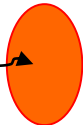
$$\sigma = \frac{3\lambda^2}{2\pi} \frac{1}{1+I/I_s + 4\delta_L^2/\Gamma^2} \quad \sigma_0 = \frac{3\lambda^2}{2\pi} \quad \text{at } \delta_L = 0 \text{ and } I \ll I_s$$

Cross section

Classical
Cross section $\Rightarrow \sigma = \frac{3\lambda^2}{2\pi} \frac{1}{1+(2\delta/\gamma)^2}$

$$I_s = \frac{\pi\hbar c}{3\lambda^3} \Gamma = \frac{\hbar\omega}{\sigma_0(2/\Gamma)}$$

At the saturation intensity, the energy of just one photon flows through the resonant absorption cross section during the transverse coherence time

$\hbar\omega$  $\sigma_0 = \frac{3\lambda^2}{2\pi}$
during $\frac{2}{\Gamma}$

Rate Equation Approximation

Adiabatic elimination of coherences

Transformation of optical Bloch equations into Rate Equations

When elastic collisions dephase the atomic dipole

Collision
term

$$\dot{\sigma}_{ge} = -i\delta_L \sigma_{ge} - i\frac{\Omega_1}{2}(\sigma_{ee} - \sigma_{gg}) - \frac{\Gamma}{2}\sigma_{ge} - \gamma\sigma_{ge}$$

Coherences evolve much more rapidly than the populations

$$\sigma_{ge}^{\text{st}} = \frac{-i(\Omega_1/2)(\sigma_{ee} - \sigma_{gg})}{[\gamma + (\Gamma/2)] + i\delta_L} \rightarrow \text{Rate Equation Approximation}$$

$$\dot{\sigma}_{ee} = -\Gamma'(\sigma_{ee} - \sigma_{gg}) - \Gamma\sigma_{ee}$$

Rate Equations

$$\dot{\sigma}_{gg} = -\Gamma'(\sigma_{gg} - \sigma_{ee}) + \Gamma\sigma_{ee}$$

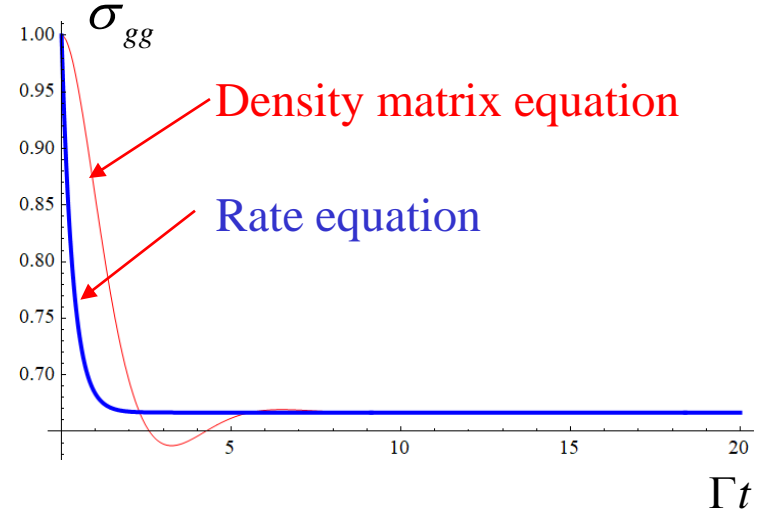
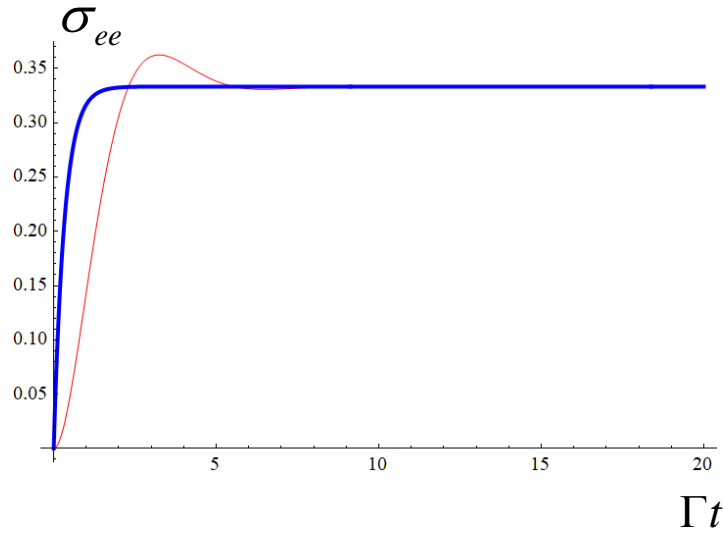
$$\Gamma' = \frac{\Omega_1^2}{2} \frac{\gamma + (\Gamma/2)}{[\gamma + (\Gamma/2)]^2 + \delta_L^2} \Rightarrow \Gamma' = \frac{\Gamma}{2} s \quad s = \frac{\Omega_1^2/2}{\delta^2 + \Gamma^2/4} = \frac{s_0}{1 + 4\delta_L^2/\Gamma^2}$$

Comparison

$$\gamma_t = \frac{\Gamma}{2}, \Omega_1 = \Gamma$$

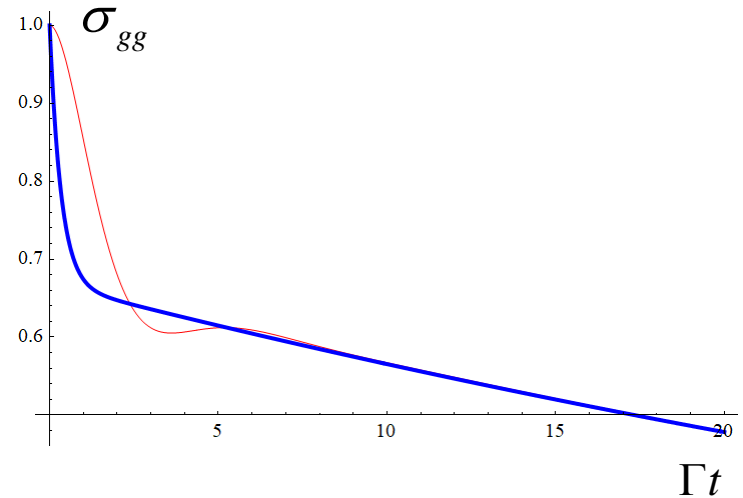
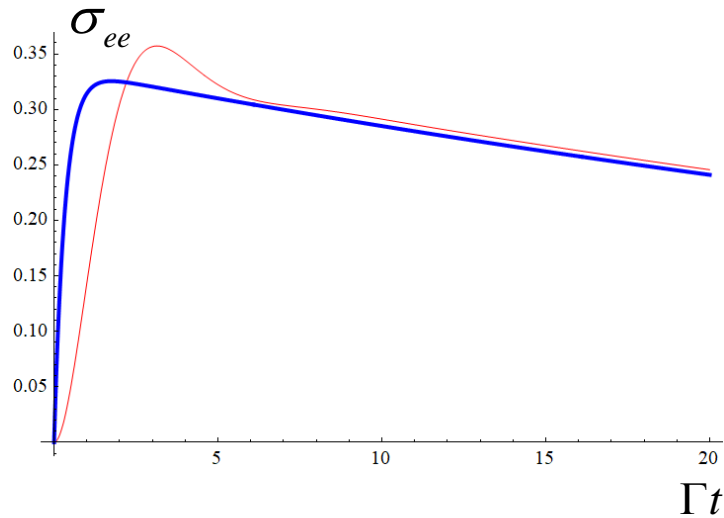
$$\Gamma_a = \Gamma$$

(closed)



$$\Gamma_a = 0.95\Gamma$$

(open)



Susceptibility

$$\begin{aligned}
 \langle \vec{d} \rangle &= \text{Tr}(\rho \vec{d}) = \vec{d}_{ge}(\rho_{ge} + \rho_{eg}) \\
 &= \vec{d}_{ge}(\sigma_{ge} e^{i\omega_L t} + \sigma_{eg} e^{-i\omega_L t}) \\
 &= 2\vec{d}_{ge}(u \cos \omega_L t - v \sin \omega_L t) \\
 &= 2\vec{d}_{ge} \text{Re}[(u - iv)e^{-i\omega t}] \\
 &= 2\vec{d}_{ge} \text{Re}[\sigma_{eg} e^{-i\omega t}]
 \end{aligned}$$

$$\vec{d}_{ge} // \vec{E}_0$$

Polarization

$$\begin{aligned}
 P &= \frac{N}{V} \langle d \rangle = \frac{N}{V} 2d_{ge} \text{Re}[\sigma_{eg} e^{-i\omega t}] \\
 &= \text{Re}[\varepsilon_0 \chi E_0 e^{-i\omega t}]
 \end{aligned}$$

$$\Rightarrow \chi = -\frac{N}{V} \frac{2d_{eg}}{\varepsilon_0 E_0} \sigma_{eg}$$

$$\begin{aligned}
 \frac{2d_{eg}}{\varepsilon_0 E_0} &= -\frac{2d_{eg}}{\varepsilon_0} \frac{d_{eg}}{\hbar \Omega_1} \quad \leftarrow \Omega_1 = -\frac{d \cdot E}{\hbar} \\
 &= -\frac{2}{\varepsilon_0 \hbar \Omega_1} d_{eg}^2 \quad \leftarrow \Gamma = \frac{1}{3\pi \varepsilon_0} \frac{\omega_0^2}{\hbar c^3} d_{eg}^2 \\
 &= -\frac{2}{\varepsilon_0 \hbar \Omega_1} \frac{\Gamma}{\omega_0^3} 3\pi \varepsilon_0 \hbar c^3 \\
 &= -\frac{2\Gamma}{\Omega_1} 3\pi c^3 \left(\frac{\lambda}{2\pi c} \right)^3 = -\frac{3\lambda^3}{4\pi^2} \frac{\Gamma}{\Omega_1}
 \end{aligned}$$

susceptibility

$$\chi = -\frac{N}{V} \frac{3\lambda^3}{4\pi^2} \frac{\Gamma}{\Omega_1} \sigma_{eg}$$

Susceptibility

susceptibility

$$\begin{aligned}\chi &= -\frac{N}{V} \frac{3\lambda^3}{4\pi^2} \frac{\Gamma}{\Omega_1} \sigma_{eg} = -\frac{N}{V} \frac{3\lambda^3}{4\pi^2} \frac{\Gamma}{\Omega_1} (u_{\text{st}} - i v_{\text{st}}) \\ &= -\frac{N}{V} \frac{3\lambda^3}{4\pi^2} \frac{\Gamma}{2} \frac{\delta_L - i \frac{\Gamma}{2}}{\delta_L^2 + (\Gamma^2/4) + (\Omega_1^2/2)}\end{aligned}$$

Refractive index

$$n' \cong 1 + \frac{1}{2} \text{Re}(\chi) = 1 - \frac{N}{V} \frac{3\lambda^3}{4\pi^2} \frac{2\delta/\Gamma}{1 + (I/I_s) + (2\delta/\Gamma)^2}$$

Absorption coefficient

$$\alpha \cong k \text{Im}(\chi) = \frac{N}{V} \frac{3\lambda^2}{2\pi} \frac{1}{1 + (I/I_s) + (2\delta/\Gamma)^2} = \frac{N}{V} \sigma$$

Contents

- **Two-Level Atoms**

- One field (Optical Bloch equations)
- **One field (Linear absorption)**
- One field (Analytical solutions to Optical Bloch equations)
- Two fields (Susceptibility)
- Two fields (Polarization spectroscopy for the $F_g=0 \rightarrow F_e=1$ transition)

- **Multi-Level Atoms**

- Density matrix equations
- Electromagnetically induced transparency and absorption
- Wave mixing and rate equations
- Elliptic polarization and arbitrary quantization axis

Maxwell Boltzmann Distribution

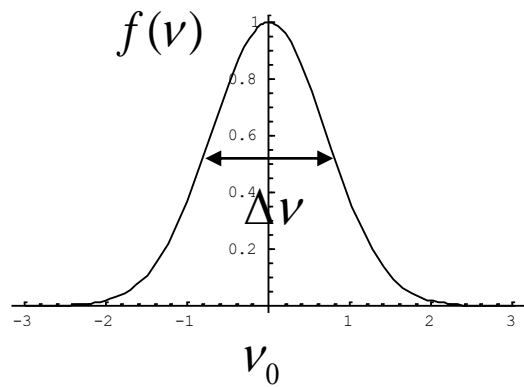
Maxwell-Boltzmann velocity distribution

$$f_D(\vec{v})dv_x dv_y dv_z = \frac{1}{(\sqrt{\pi}v_{\text{mp}})^3} e^{-(v/v_{\text{mp}})^2} dv_x dv_y dv_z$$

$$v_{\text{mp}} = \sqrt{\frac{2k_B T}{M}} \quad \text{Most Probable Velocity}$$

1D Maxwell-Boltzmann velocity distribution

$$f_z(v_z)dv_z = \frac{1}{\sqrt{\pi}v_{\text{mp}}} e^{-(v_z/v_{\text{mp}})^2} dv_z$$



FWHM (Full Width at Half Maximum)

$$\frac{\Delta v}{v_0} = \frac{2\sqrt{2\ln 2}}{c} \sqrt{\frac{k_B T}{M}} = 2\sqrt{\ln 2} \frac{u}{c}$$

^{133}Cs Atom, at Room Temp. $\Delta\nu = 380 \text{ MHz}$

Natural Linewidth $\sim 6 \text{ MHz}$

Doppler Effects

In the rest frame of an atom moving at \vec{v}

The frequency experienced by the atom $\omega \Rightarrow \omega - \vec{k} \cdot \vec{v}$

Detuning $\delta = \omega - \omega_0 \Rightarrow \omega - \vec{k} \cdot \vec{v} - \omega_0 = \delta - \vec{k} \cdot \vec{v}$

Longitudinal velocity v_z $\vec{k} = k\hat{z}$

Doppler averaging over MB distribution

$$\rho(\delta) \rightarrow \int_{-\infty}^{\infty} dv_z \frac{1}{\sqrt{\pi} v_{\text{mp}}} e^{-(v_z / v_{\text{mp}})^2} \rho(\delta - kv_z)$$

Susceptibility Averaged over Velocity Distribution

Two Level Atom

$$\chi = -\frac{N}{V} \frac{3\lambda^3}{4\pi^2} \frac{\Gamma}{\Omega_1} \sigma_{eg} \quad \leftarrow \quad \sigma_{eg} = \frac{(\Omega_1/2)(\delta_1 - i\gamma_t)}{\delta_1^2 + \gamma_t^2 + (\Omega_1^2\gamma_t/\Gamma)}, \quad \delta_1 = \delta - kv$$

$$\int_{-\infty}^{\infty} \frac{e^{-(v/v_{mp})^2}}{\sqrt{\pi}v_{mp}} \chi(v) dv$$

$$\begin{aligned} & \int_{-\infty}^{\infty} \frac{e^{-(v/v_{mp})^2}}{\sqrt{\pi}v_{mp}} \sigma_{eg} dv \\ &= \int_{-\infty}^{\infty} \frac{e^{-(v/v_{mp})^2}}{\sqrt{\pi}v_{mp}} \frac{(\Omega_1/2)(\delta - kv - i\gamma_t)}{(\delta - kv)^2 + \gamma_t^2 + (\Omega_1^2\gamma_t/\Gamma)} dv \\ &= \int_{-\infty}^{\infty} \left(\frac{b_1}{y - z_1} + \frac{b_2}{y - z_2} \right) e^{-y^2} dy \end{aligned}$$

Use $\int_{-\infty}^{\infty} \frac{e^{-y^2}}{y - z} dy = is\pi e^{-z^2} \text{Erfc}(-isz)$

$$s = \text{sign} [\text{Im}(z)]$$

$$\begin{aligned} z_1 &= \frac{\delta}{kv_{mp}} + i \frac{\gamma_t}{kv_{mp}} \sqrt{1 + s_0} \\ z_2 &= \frac{\delta}{kv_{mp}} - i \frac{\gamma_t}{kv_{mp}} \sqrt{1 + s_0} \\ b_1 &= -\frac{(1 + \sqrt{1 + s_0})\Omega_1}{4\sqrt{\pi}kv_{mp}\sqrt{1 + s_0}} \\ b_2 &= \frac{(1 - \sqrt{1 + s_0})\Omega_1}{4\sqrt{\pi}kv_{mp}\sqrt{1 + s_0}} \end{aligned}$$

$$s_0 = \frac{\Omega_1^2}{\Gamma_1\gamma_t}$$

Linear Absorption

$$\frac{\chi}{C_0} = \frac{i(1+\sqrt{1+s_0})}{2\sqrt{1+s_0}} e^{-z_1^2} \text{Erfc}(-i z_1) + \frac{i(1-\sqrt{1+s_0})}{2\sqrt{1+s_0}} e^{-z_2^2} \text{Erfc}(i z_2)$$

$$z_1 = \frac{\delta}{kv_{\text{mp}}} + i \frac{\gamma_t}{kv_{\text{mp}}} \sqrt{1+s_0}, \quad z_2 = \frac{\delta}{kv_{\text{mp}}} - i \frac{\gamma_t}{kv_{\text{mp}}} \sqrt{1+s_0}, \quad s_0 = \frac{\Omega_1^2}{\Gamma_1 \gamma_t}, \quad C_0 = \frac{3\lambda^3}{8\pi^{3/2}} \frac{\Gamma N_{\text{at}}}{kv_{\text{mp}}}$$

In the Doppler-broadened limit $\frac{\gamma_t}{kv_{\text{mp}}} \ll 1$ ^{87}Rb $v_{\text{mp}} = 236.6 \text{ m/s} \gg \frac{\Gamma}{k} = 4.68 \text{ m/s}$

Use $\text{Erfc}\left(\pm i \frac{\delta}{kv_{\text{mp}}}\right) = 1 \mp i \text{Erfi}\left(\frac{\delta}{kv_{\text{mp}}}\right)$

$$\frac{\chi}{C_0} = e^{-(\delta/kv_{\text{mp}})^2} \left(-\text{Erfi}\left(\frac{\delta}{kv_{\text{mp}}}\right) + \frac{i}{\sqrt{1+s_0}} \right)$$

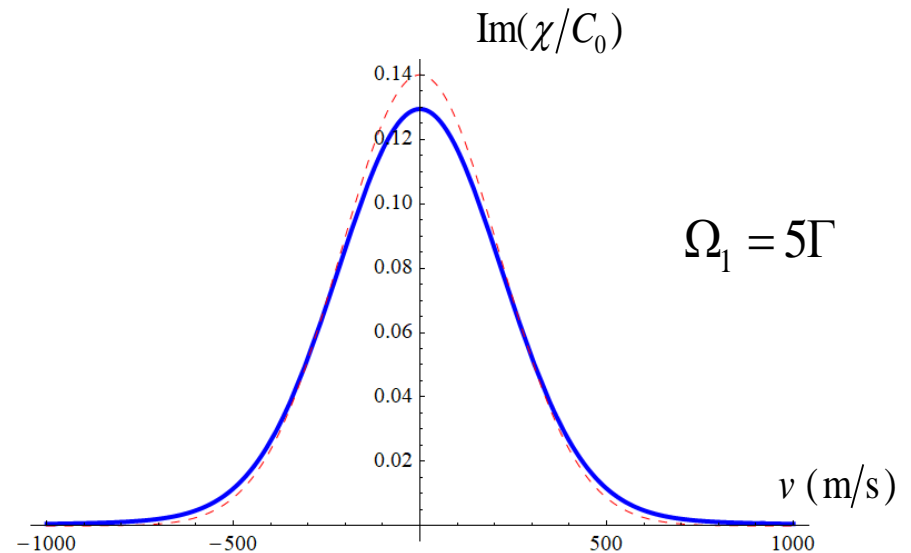
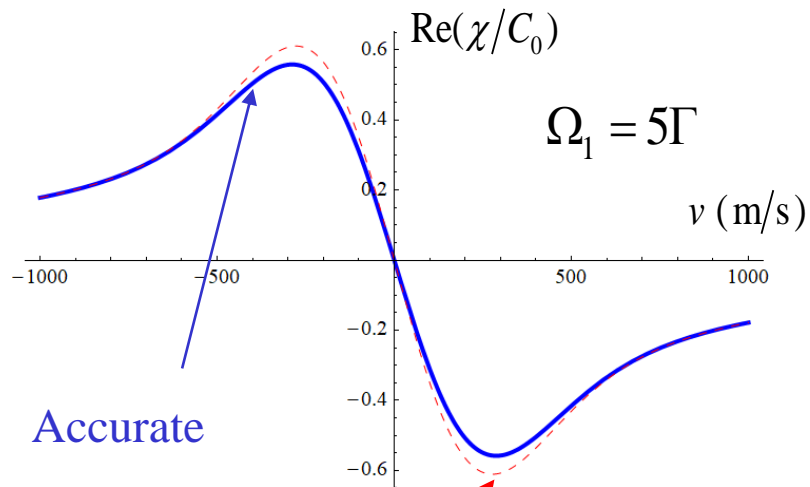
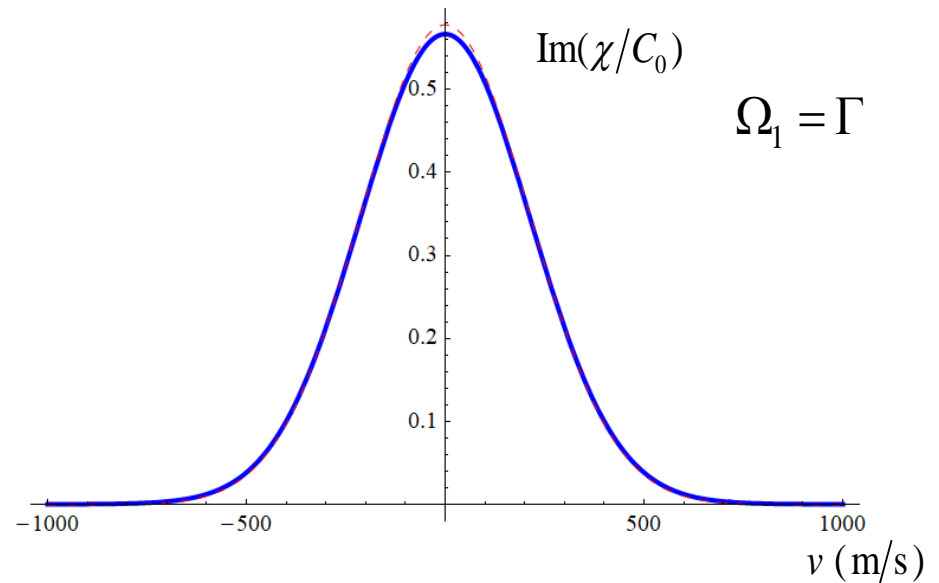
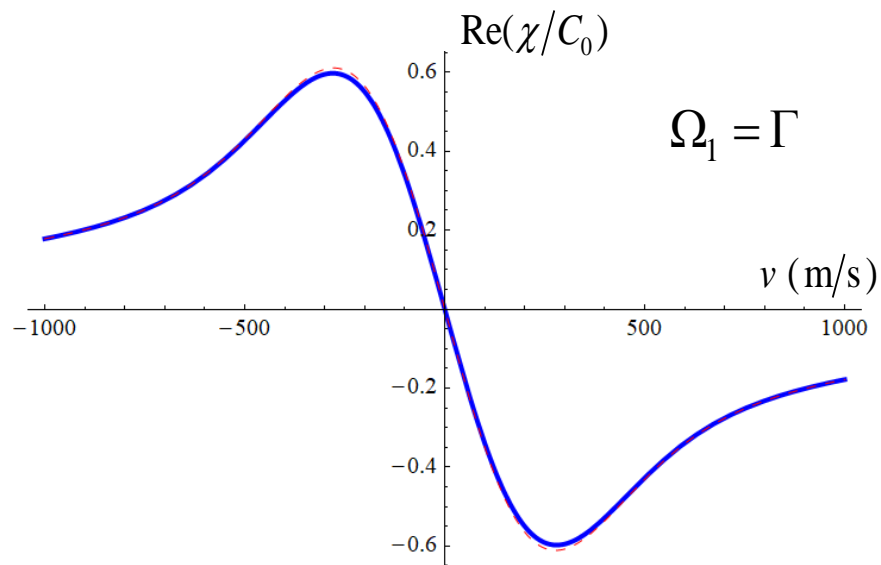


$$n = 1 - \frac{C_0}{2} e^{-(\delta/kv_{\text{mp}})^2} \text{Erfi}\left(\frac{\delta}{kv_{\text{mp}}}\right)$$

$$\alpha = \frac{3\lambda^3}{2\pi} \cdot \frac{N_{\text{at}} \Gamma}{4\sqrt{\pi} kv_{\text{mp}}} \cdot \frac{1}{\sqrt{1+s_0}} e^{-(\delta/kv_{\text{mp}})^2}$$

→ saturation

Linear Dispersion and Absorption



Doppler-broadened limit

Contents

- **Two-Level Atoms**

- One field (Optical Bloch equations)
- One field (Linear absorption)
- **One field (Analytical solutions to Optical Bloch equations)**
- Two fields (Susceptibility)
- Two fields (Polarization spectroscopy for the $F_g=0 \rightarrow F_e=1$ transition)

- **Multi-Level Atoms**

- Density matrix equations
- Electromagnetically induced transparency and absorption
- Wave mixing and rate equations
- Elliptic polarization and arbitrary quantization axis

Analytical Solutions to OBEs

- Analytic Solutions to the Optical Bloch Equations

$$\dot{u} = -\gamma_t u + \delta_1 v$$

$$\dot{v} = -\gamma_t v - \delta_1 u + \frac{\Omega_1}{2} w$$

$$\dot{w} = -\gamma w - 2\Omega_1 v + \gamma$$

- Torrey's solution

H. C. Torrey, Phys. Rev. 76, 1059 (1949).

H. R. Noh, W. Jhe, Opt. Commun. 283, 2353 (2010).

Optical Bloch Equations

Optical Bloch Equations

$$\dot{u} = -\gamma_t u + \delta_1 v$$

$$\dot{v} = -\gamma_t v - \delta_1 u + \frac{\Omega_1}{2} w$$

$$\dot{w} = -\gamma w - 2\Omega_1 v + \gamma$$

Trial solution $e^{\lambda t}$

Secular equation for the solutions

$$(\lambda + \gamma) \left[(\lambda + \gamma_t)^2 + \delta_1^2 \right] + (\lambda + \gamma_t) \Omega_1^2 = 0$$

Three solutions

$$\lambda_1 = -a, \quad \lambda_{2(3)} = -p + (-)iq$$

$$(i) \quad R^2 + Q^3 > 0$$

a, p, q :
real and positive

Steady-State Solutions

$$u_{\text{st}} = \frac{\delta_1 \Omega_1 / 2}{\delta_1^2 + \gamma_t \left(\gamma_t + \Omega_1^2 / \gamma \right)},$$

$$v_{\text{st}} = \frac{\gamma_t \Omega_1 / 2}{\delta_1^2 + \gamma_t \left(\gamma_t + \Omega_1^2 / \gamma \right)},$$

$$w_{\text{st}} = \frac{\delta_1^2 + \gamma_t^2}{\delta_1^2 + \gamma_t \left(\gamma_t + \Omega_1^2 / \gamma \right)}$$

$$a = \alpha - (S_+ + S_-),$$

$$p = \alpha + \frac{1}{2}(S_+ + S_-), \quad q = \frac{\sqrt{3}}{2}(S_+ - S_-),$$

$$\alpha = \frac{2\gamma_t + \gamma}{3}, \quad S_{\pm} = \left(R \pm \sqrt{R^2 + Q^3} \right)^{1/3},$$

$$Q = \frac{1}{3}(\delta_1^2 + \Omega_1^2) - \frac{1}{9}(\gamma_t - \gamma)^2,$$

$$R = \frac{1}{54}(\gamma_t - \gamma) \left[9(2\delta_1^2 - \Omega_1^2) + 2(\gamma_t - \gamma)^2 \right]$$

Map for the Solutions

$$(ii) R^2 + Q^3 < 0$$

$$\lambda_1 = -a, \quad \lambda_{2(3)} = -p + (-)s$$

$$a = \alpha - 2c, \quad p = \alpha + c,$$

$$c = \sqrt{-Q} \cos \phi, \quad s = \sqrt{-3Q} \sin \phi,$$

$$\phi = \frac{1}{3} \cos^{-1} \left[(-Q)^{-3/2} R \right]$$

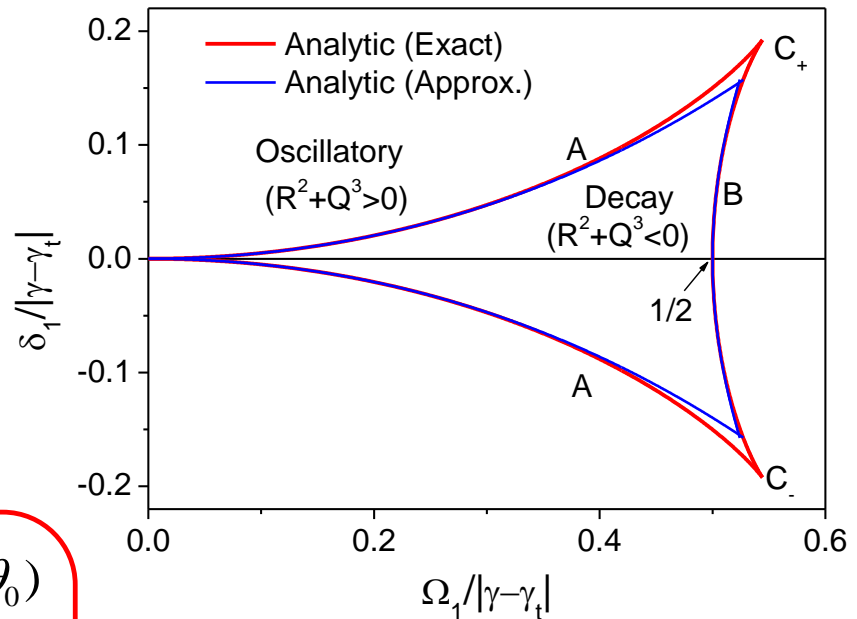
$$\delta_2^2 = -\frac{1}{3} (2 + 3\Omega_2^2) + \frac{2}{3} \sqrt{1 + 27\Omega_2^2} \cos(\theta + \theta_0)$$

$$\theta = \frac{1}{3} \cos^{-1} \left[\frac{8 - 27\Omega_2^2 (20 + 27\Omega_2^2)}{8(1 + 27\Omega_2^2)^{3/2}} \right]$$

$$\theta_0 = \begin{cases} 0 & \text{for A} \\ 4\pi/3 & \text{for B} \end{cases}$$

$$\delta_2 = \frac{\delta_1}{|\gamma - \gamma_t|},$$

$$\Omega_2 = \frac{\Omega_1}{|\gamma - \gamma_t|}$$



Approximate formula for A and B

$$\delta_2^2 = \begin{cases} \Omega_2^4 (1 + \Omega_2^2) & \text{for A} \\ \Omega_2^2 - (1/4) & \text{for B} \end{cases}$$

$$C_{\pm} \quad \Omega_1 = (2/3)^{3/2} |\gamma - \gamma_t|$$

$$\delta_1 = \pm 3^{-3/2} |\gamma - \gamma_t|$$

Solutions (Outside the Boundary)

$$(i) R^2 + Q^3 > 0$$

$$\eta = Ae^{-at} + \left[B \cos qt + \frac{C}{q} \sin qt \right] e^{-pt} + \eta_s$$

$$A = \frac{1}{\Delta} \left[\eta_2 + 2\eta_1 p + (p^2 + q^2)(\eta_0 - \eta_s) \right]$$

$$B = \frac{1}{\Delta} \left[a(a - 2p)(\eta_0 - \eta_s) - (\eta_2 + 2\eta_1 p) \right]$$

$$C = \frac{1}{\Delta} \left[\eta_2(a - p) + \eta_1(a^2 - p^2 + q^2) + a(ap - p^2 + q^2)(\eta_0 - \eta_s) \right]$$

$$\Delta = (a - p)^2 + q^2$$

$$\eta_i \equiv \frac{d^i \eta}{dt^i}(0)$$

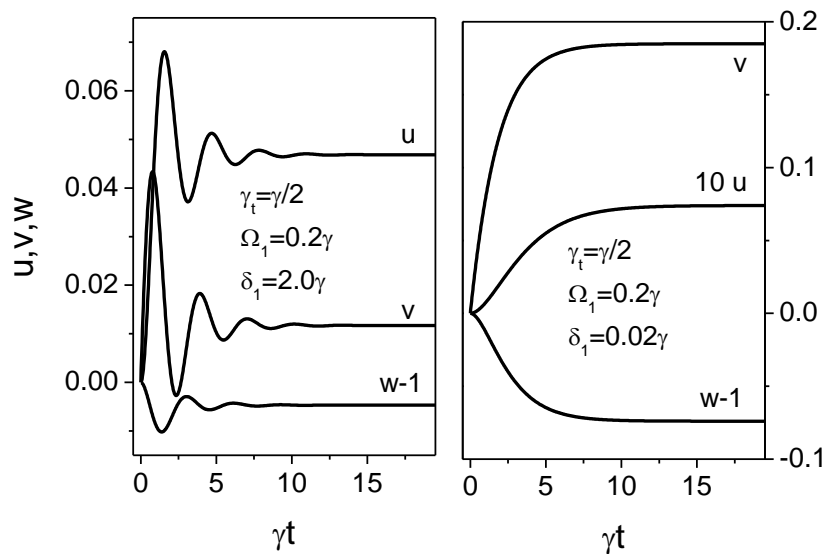
$$\begin{aligned} u_0 &= 0, & u_1 &= 0, & u_2 &= \delta_1 \Omega_1 / 2, \\ v_0 &= 0, & v_1 &= \Omega_1 / 2, & v_2 &= -\gamma_t \Omega_1 / 2, \\ w_0 &= 1, & w_1 &= 0, & w_2 &= -\Omega_1^2, \end{aligned}$$

Solutions (Inside the Boundary)

(ii) $R^2 + Q^3 < 0$

$$\eta = Ae^{-at} + \left[B \cosh st + \frac{C}{s} \sinh st \right] e^{-pt} + \eta_s$$

$A, B, C, \Delta \quad q^2 \rightarrow -s^2$



Typical Results

Contents

- **Two-Level Atoms**

- One field (Optical Bloch equations)
- One field (Linear absorption)
- One field (Analytical solutions to Optical Bloch equations)
- **Two fields (Susceptibility)**
- Two fields (Polarization spectroscopy for the $F_g=0 \rightarrow F_e=1$ transition)

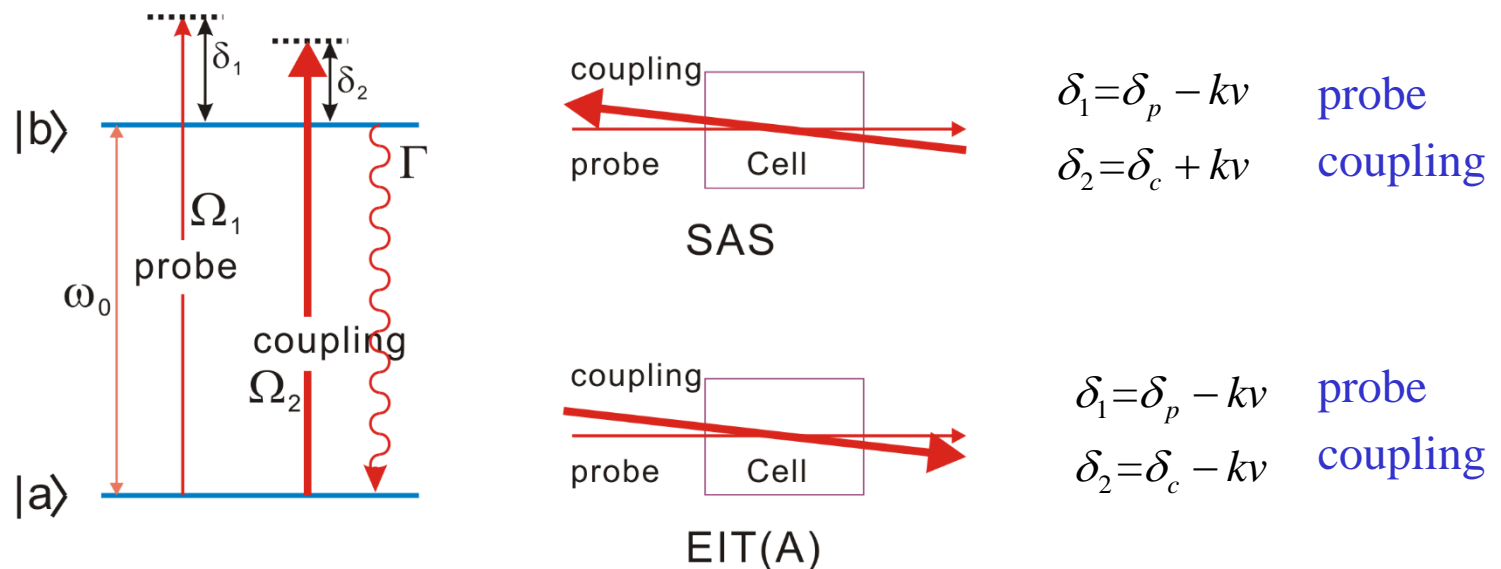
- **Multi-Level Atoms**

- Density matrix equations
- Electromagnetically induced transparency and absorption
- Wave mixing and rate equations
- Elliptic polarization and arbitrary quantization axis

Two Fields: Susceptibility for Two-level Atoms

Two-level atoms: two laser frequencies

- Accurate analytical formula for susceptibility for Doppler-broadened two-level atoms
- Weak probe beam and arbitrary pump beam
- Hole-burning and coherence contributions
- Doppler-broadened limit



H. R. Noh, J. Opt. Soc. Am. B **33**, 308 (2016). SAS

H. R. Noh, J. Opt. Soc. Am. B **33**, 1523 (2016). PS

Density Matrix Equations

Density matrix equation

$$\dot{\rho} = -\frac{i}{\hbar}[H, \rho] + (\dot{\rho})_{\text{sp}}$$

$$H = \hbar \begin{pmatrix} \omega_0 & \frac{\Omega_1}{2} e^{-i\omega_1 t} + \frac{\Omega_2}{2} e^{-i\omega_2 t} \\ \frac{\Omega_1}{2} e^{i\omega_1 t} + \frac{\Omega_2}{2} e^{i\omega_2 t} & 0 \end{pmatrix}$$

$$(\dot{\rho})_{\text{sp}} = \begin{pmatrix} -\Gamma \rho_{bb} & -\gamma_t \rho_{ba} \\ -\gamma_t \rho_{ab} & \Gamma \rho_{bb} \end{pmatrix}$$

Transformation

$$\sigma = U \rho U^\dagger \quad U = \begin{pmatrix} e^{i\omega_2 t} & 0 \\ 0 & 1 \end{pmatrix} \quad H' = U H U^\dagger + i \frac{\partial U}{\partial t} U^\dagger$$

$$\dot{\sigma} = -\frac{i}{\hbar}[H', \sigma] + (\dot{\sigma})_{\text{sp}}$$

Explicit occurrence of
time dependent terms



$$H' = \hbar \begin{pmatrix} -\delta_2 & \frac{\Omega_1}{2} e^{-i\delta_d t} + \frac{\Omega_2}{2} \\ \frac{\Omega_1}{2} e^{i\delta_d t} + \frac{\Omega_2}{2} & 0 \end{pmatrix}$$

$$(\dot{\sigma})_{\text{sp}} = \begin{pmatrix} -\Gamma \sigma_{bb} & -\gamma_t \sigma_{ba} \\ -\gamma_t \sigma_{ab} & \Gamma \sigma_{bb} \end{pmatrix} \quad \sigma = \begin{pmatrix} \sigma_{bb} & \sigma_{ba} \\ \sigma_{ab} & \sigma_{aa} \end{pmatrix}$$

$$\delta_d = \delta_1 - \delta_2 = -2kv$$

$$\delta_1 = \delta - kv$$

$$\delta_2 = \delta + kv$$

or

$$\delta_d = \delta_1 - \delta_2 = 2kv$$

$$\delta_1 = \delta + kv$$

$$\delta_2 = \delta - kv$$

Calculation of Oscillation Frequencies

Finding non-vanishing density matrix elements

Involving frequencies: ω_1, ω_2 (Probe, coupling)

$$\delta_d = \omega_1 - \omega_2 = \delta_1 - \delta_2$$

Involving frequencies relative to ω_2 : $\delta_d, 0$

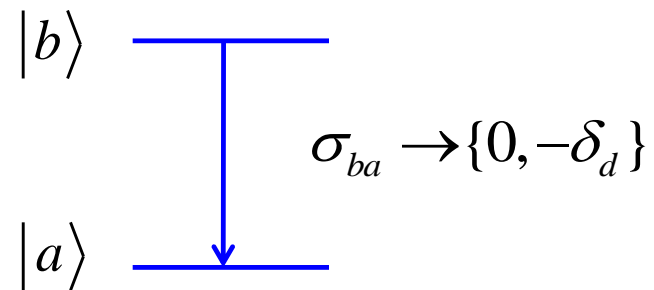
Emission process: $\{0, -\delta_d\}$

Absorption process: $\{0, \delta_d\}$

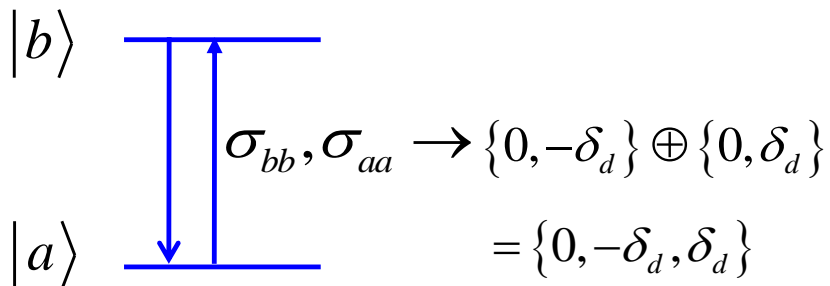
Zero-photon interaction

$\sigma_{bb}, \sigma_{aa} \rightarrow$ No time dependence

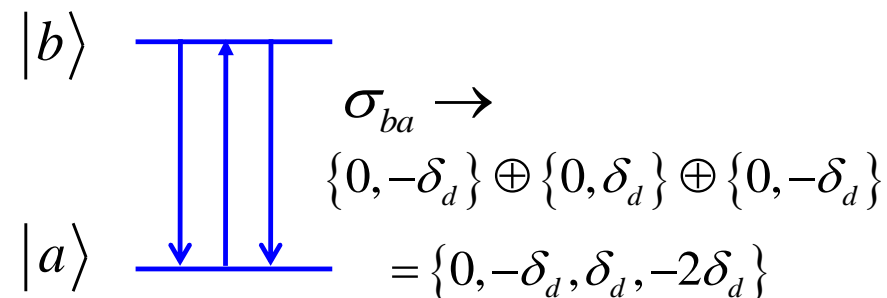
One-photon interaction



Two-photon interaction

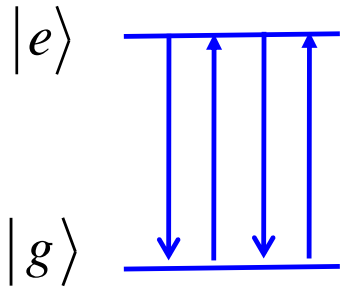


Three-photon interaction



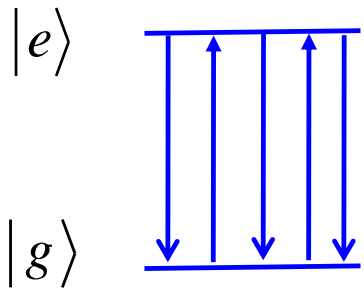
Calculation of Oscillation Frequencies

Four-photon interaction



$$\begin{aligned}\sigma_{bb}, \sigma_{aa} &\rightarrow \{0, -\delta_d\} \oplus \{0, \delta_d\} \oplus \{0, -\delta_d\} \oplus \{0, \delta_d\} \\ &= \{0, -\delta_d, \delta_d, -2\delta_d, 2\delta_d\}\end{aligned}$$

Five-photon interaction



$$\begin{aligned}\sigma_{ba} &\rightarrow \{0, -\delta_d\} \oplus \{0, \delta_d\} \oplus \{0, -\delta_d\} \oplus \{0, \delta_d\} \oplus \{0, -\delta_d\} \\ &= \{0, -\delta_d, \delta_d, -2\delta_d, 2\delta_d, -3\delta_d\}\end{aligned}$$

Density Matrix Elements

Considering up to three photon interactions

$$\sigma_{bb}, \sigma_{aa} \rightarrow \{0, -\delta_d, \delta_d\}$$

$$\sigma_{ba} \rightarrow \{0, -\delta_d, \delta_d, -2\delta_d\}$$

$$\sigma_{ab} \rightarrow \{0, \delta_d, -\delta_d, 2\delta_d\}$$

Populations

$$\sigma_{bb} = q_1 + (q_2 + iq_3)e^{-i\delta_d t} + (q_2 - iq_3)e^{i\delta_d t}$$

$$\sigma_{aa} = p_1 + (p_2 + ip_3)e^{-i\delta_d t} + (p_2 - ip_3)e^{i\delta_d t}$$

Coherence

$$\sigma_{ba} = (r_1 + is_1) + (r_2 + is_2)e^{-i\delta_d t} + (r_3 + is_3)e^{i\delta_d t} + (r_4 + is_4)e^{-2i\delta_d t}$$

$$\sigma_{ab} = \sigma_{ba}^*$$

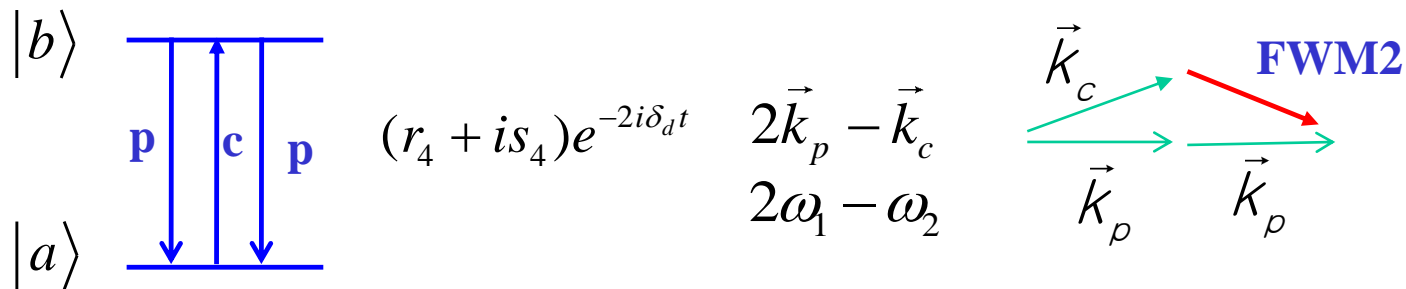
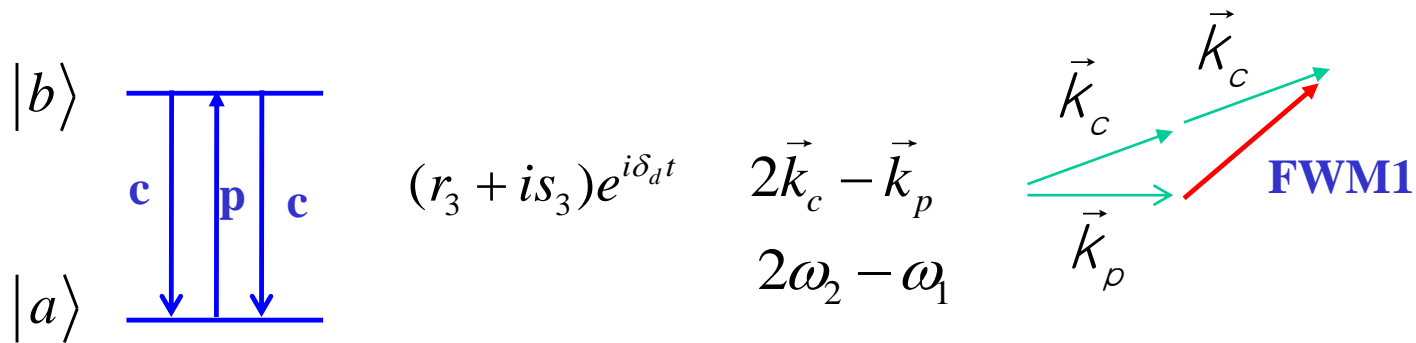
Pump absorption

Probe absorption

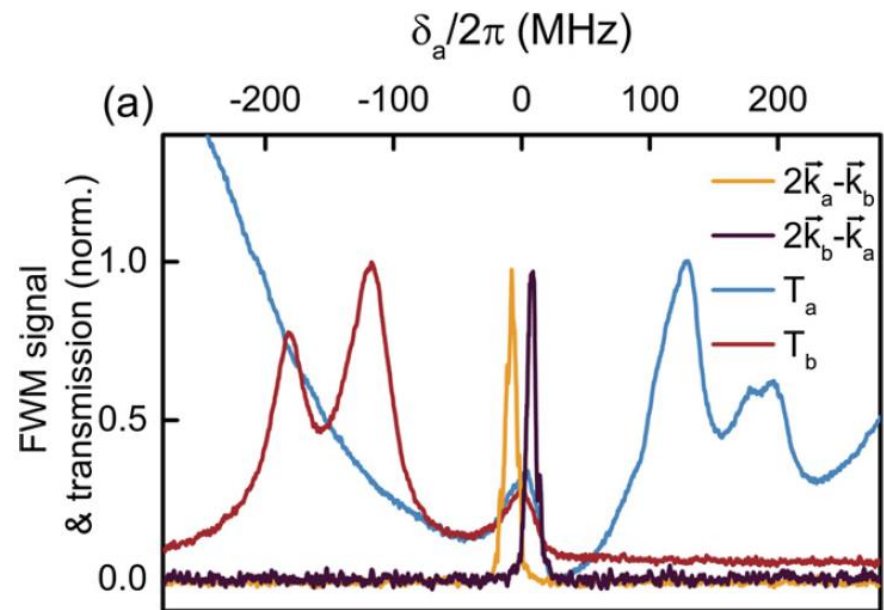
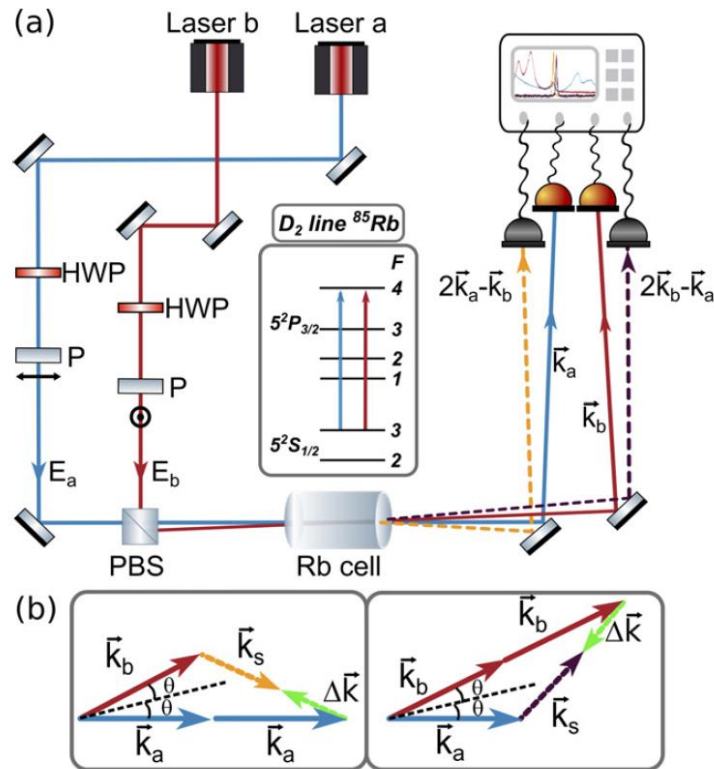
Four wave mixing 1

Four wave mixing 2

Four Wave Mixing



Four Wave Mixing



A. S. Alvarez et al., J. Phys. B: At. Mol. Opt. Phys. **54**, 045403 (2021).

Density Matrix Equations

Fourier expansion of density matrix elements

Up to 1st order in Ω_1

$$\sigma_{ba} = (r_1 + is_1) + (r_2 + is_2)e^{-i\delta_d t} + (r_3 + is_3)e^{i\delta_d t} + (r_4 + is_4)e^{-2i\delta_d t}$$

Neglecting

Density matrix equations

$$\sigma_{ba} = z_1 + z_2 e^{-i\delta_d t} + z_3 e^{i\delta_d t}$$

$$\sigma_{ab} = z_1^* + z_2^* e^{-i\delta_d t} + z_3^* e^{i\delta_d t}$$

$$\sigma_{bb} = q + u e^{-i\delta_d t} + u^* e^{i\delta_d t}$$

$$\sigma_{aa} = p - u e^{-i\delta_d t} - u^* e^{i\delta_d t}$$

$$\dot{z}_1 = i\Delta_1 z_1 + i\Omega_1 u^* + i\frac{\Omega_2}{2}(q - p),$$

$$\dot{z}_2 = i\Delta_2 z_2 + i\Omega_2 u + i\frac{\Omega_1}{2}(q - p),$$

$$\dot{z}_3 = i\Delta_3 z_3 + i\Omega_2 u^*,$$

$$\dot{u} = i\Delta_4 u - i\frac{\Omega_1}{2}z_1^* + i\frac{\Omega_2}{2}z_2 - i\frac{\Omega_2}{2}z_3^*,$$

$$\dot{q} = -\Gamma q - \Omega_1 \text{Im}(z_2) - \Omega_2 \text{Im}(z_1),$$

$$\delta_d = \delta_1 - \delta_2 = 2kv$$

where $\delta_1 = \delta + kv$

$$\delta_2 = \delta - kv$$

$$\Delta_1 = \delta_2 + i\gamma_\nu \quad \Delta_2 = \delta_2 + \delta_d + i\gamma_\nu$$

$$\Delta_3 = \delta_2 - \delta_d + i\gamma_\nu \quad \Delta_4 = \delta_d + i\Gamma.$$

Solving in the steady-state regime

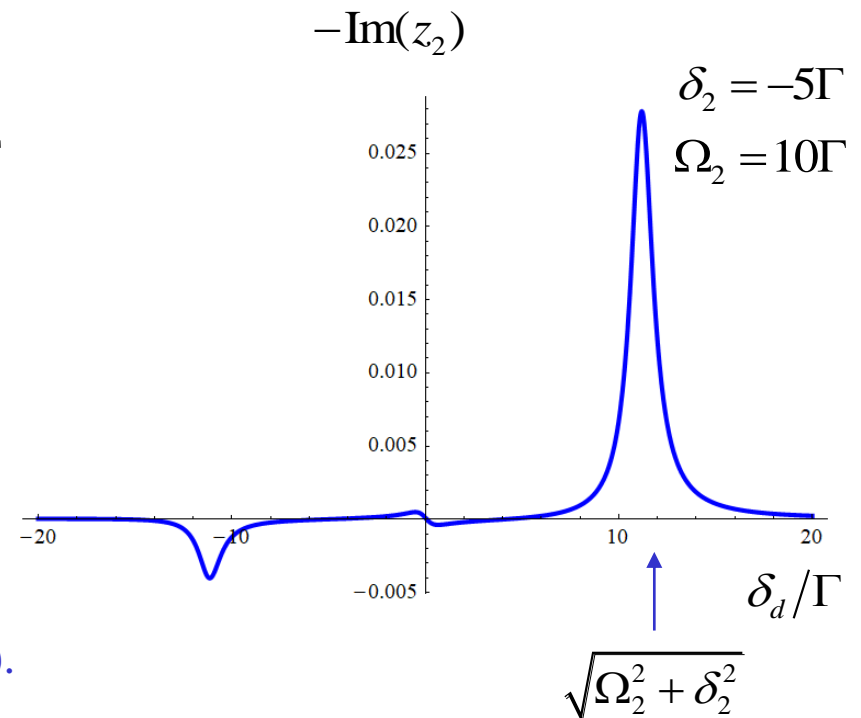
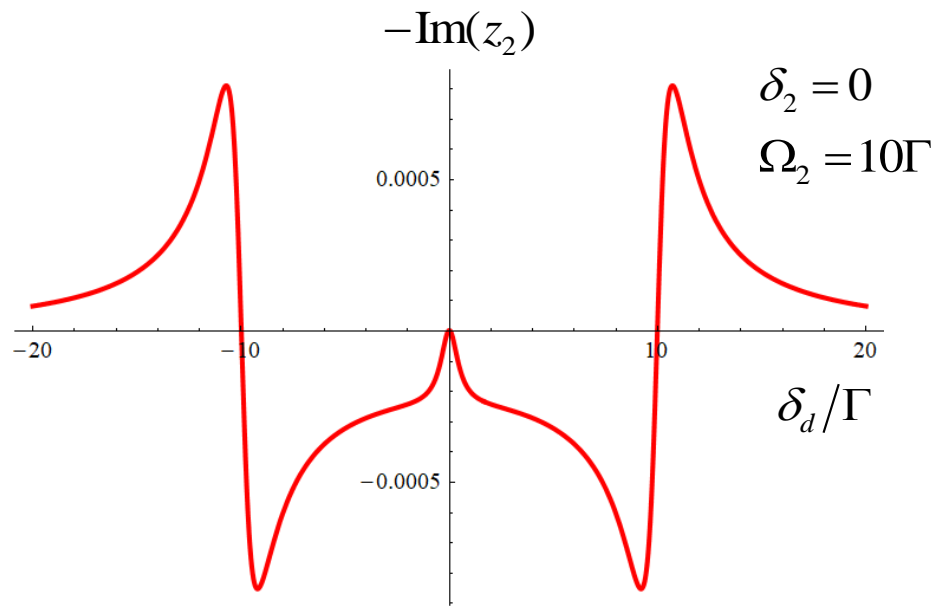
Probe Absorption

$$z_2 = \frac{\frac{\Omega_1}{2} \left((\delta_d + i\Gamma) (\delta_2 - \delta_d - i\gamma_t) + \frac{\delta_d \Omega_2^2}{2(\delta_d - i\gamma_t)} \right)}{(\delta_2 - \delta_d - i\gamma_t) (\delta_2 + \delta_d + i\gamma_t) (\delta_d + i\Gamma) + (\delta_d + i\gamma_t) \Omega_2^2} \times (p - q)$$

$$p - q = \frac{\delta_2^2 + \gamma_t^2}{\delta_2^2 + \gamma_t^2 + \frac{\gamma_t \Omega_2^2}{\Gamma}}$$

Probe absorption
for stationary atoms

Mollow triplet



B. R. Mollow, Phys. Rev. **188**, 1969 (1969).

B. R. Mollow, Phys. Rev. A **5**, 2217 (1972).

Dressed States

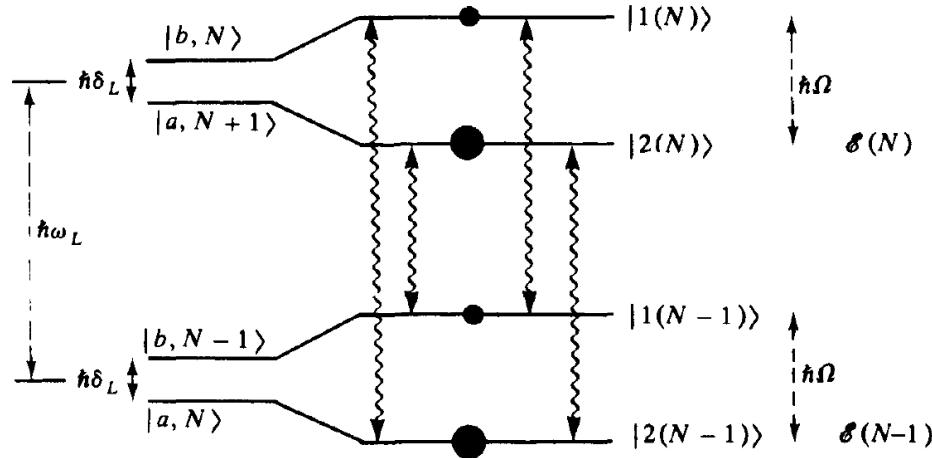
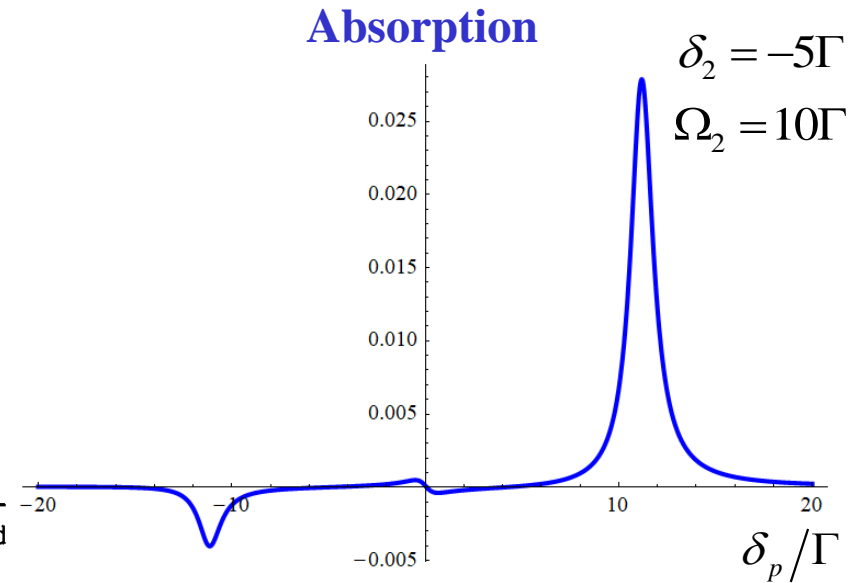


Figure 11. Dressed levels and quasi-steady-state populations of the atom + photons ω_L system. The wavy arrows indicate the transitions probed by the second laser beam.



C. Cohen-Tannoudji,

“Atom-Photon Interactions”

p. 443

Susceptibility

$$\chi = -\frac{3\lambda^3}{4\pi^2} \cdot \frac{\Gamma N_{\text{at}}}{\Omega_1} \cdot \frac{1}{\sqrt{\pi} v_{\text{mp}}} \int_{-\infty}^{\infty} dv e^{-(v/v_{\text{mp}})^2} z_2,$$

$$\dot{z}_2 = i\Delta_2 z_2 + i\Omega_2 u + i\frac{\Omega_1}{2}(q - p).$$



$$z_2 = \boxed{\frac{\Omega_1}{2\Delta_2}} + \boxed{\frac{\Omega_1}{2\Delta_2}(p - q - 1)} - \boxed{\frac{\Omega_2}{\Delta_2}u},$$

↑ background
 ↑ hole-
burning
 ↑ coherence

$$u = \frac{(\delta_d + 2i\gamma_t)(\delta_2 - \delta_d - i\gamma_t)\Omega_1\Omega_2}{4(\delta_2 - i\gamma_t)} \times \frac{(p - q)}{(\delta_d + i\Gamma)(\delta_2 + \delta_d + i\gamma_t)(\delta_2 - \delta_d - i\gamma_t) + (\delta_d + i\gamma_t)\Omega_2^2}$$

$$p - q = \frac{\delta_2^2 + \gamma_t^2}{\delta_2^2 + \gamma_t^2 + \frac{\gamma_t\Omega_2^2}{\Gamma}}$$

$$\Delta_2 = \delta_2 + \delta_d + i\gamma_t = \delta_1 + i\gamma_t = \delta + kv + i\gamma_t$$

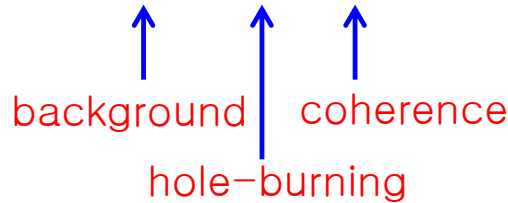
$$\delta_d = \delta_1 - \delta_2 = 2kv \quad \delta_1 = \delta + kv$$

$$\delta_2 = \delta - kv$$

Doppler Averaged Susceptibility

Doppler-averaged
Susceptibility

$$\chi = \chi_0 + \chi_1 + \chi_2$$



 background hole-burning coherence

$$\chi = -\frac{3\lambda^3}{4\pi^2} \cdot \frac{\Gamma N_{\text{at}}}{\Omega_1} \cdot \frac{1}{\sqrt{\pi} v_{\text{mp}}} \int_{-\infty}^{\infty} dv e^{-(v/v_{\text{mp}})^2} z_2,$$

1. Background

$$\chi_0 = -\frac{3\lambda^3}{4\pi^2} \cdot \frac{\Gamma N_{\text{at}}}{\Omega_1} \cdot \frac{1}{\sqrt{\pi} v_{\text{mp}}} \int_{-\infty}^{\infty} dv e^{-(v/v_{\text{mp}})^2} \frac{\Omega_1}{2(\delta - kv + i\gamma_t)}$$

Use $\int_{-\infty}^{\infty} \frac{e^{-y^2}}{y - z} dy = is\pi e^{-z^2} \text{Erfc}(-isz)$
 $s = \text{sign} [\text{Im}(z)]$

$$\chi_0 = iC_0 \exp \left[-\left(\frac{\delta + i\gamma_t}{kv_{\text{mp}}} \right)^2 \right] \text{Erfc} \left[-i \frac{\delta + i\gamma_t}{kv_{\text{mp}}} \right]$$

In the Doppler-broadened limit

$$\frac{\gamma_t}{kv_{\text{mp}}} \ll 1$$

$$\chi_0 = C_0 e^{-\delta^2/(kv_{\text{mp}})^2} \left[-\text{Erfi} \left(\frac{\delta}{kv_{\text{mp}}} \right) + i \right],$$

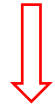
$$C_0 = \frac{3\lambda^3}{8\pi^{3/2}} \frac{\Gamma N_{\text{at}}}{kv_{\text{mp}}}$$

Hole-Burning Term

2. Hole-burning term

$$z_2 = \frac{\Omega_1}{2\Delta_2} + \boxed{\frac{\Omega_1}{2\Delta_2}(p-q-1)} - \frac{\Omega_2}{\Delta_2}u,$$

$$\chi_1 = -\frac{3\lambda^3}{4\pi^2} \cdot \frac{\Gamma N_{\text{at}}}{\Omega_1} \cdot \frac{1}{\sqrt{\pi}v_{\text{mp}}} \int_{-\infty}^{\infty} dv e^{-(v/v_{\text{mp}})^2} \frac{\Omega_1}{2(\delta - kv + i\gamma_t)} (p-q-1)$$



In the Doppler-broadened limit

$$\chi_1 = -\frac{3\lambda^3}{4\pi^2} \cdot \frac{\Gamma N_{\text{at}}}{\Omega_1} \cdot e^{-(\delta/kv_{\text{mp}})^2} \cdot \frac{1}{\sqrt{\pi}v_{\text{mp}}} \int_{-\infty}^{\infty} dv \frac{\Omega_1}{2(\delta - kv + i\gamma_t)} (p-q-1)$$

$$p-q = \frac{\delta_2^2 + \gamma_t^2}{\delta_2^2 + \gamma_t^2 + \frac{\gamma_t \Omega_2^2}{\Gamma}}$$

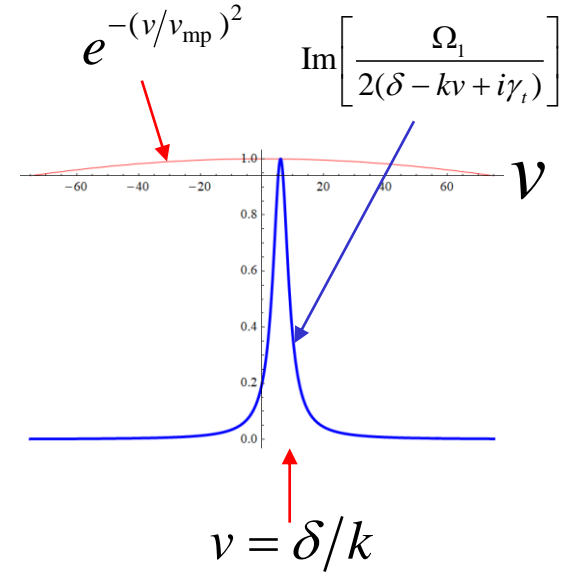


$$\frac{1}{\sqrt{\pi} v_{\text{mp}}} \int_{-\infty}^{\infty} dv \left(\frac{\Omega_1}{2(i\gamma_t + \delta_2 + \delta_1)} \left(-\frac{\gamma_t \Omega_2^2}{\Gamma(\gamma_t^2 + \delta_2^2) + \gamma_t \Omega_2^2} \right) \right) / . \delta_1 \rightarrow (\delta + x) / . \delta_2 \rightarrow (\delta - x) / . v \rightarrow (x/k) / . dv \rightarrow (dx/k) // \text{FullSimplify}$$

$$-\frac{dx \gamma_t \Omega_1 \Omega_2^2}{2 k \sqrt{\pi} v_{\text{mp}} (x + i\gamma_t + \delta) (\Gamma(\gamma_t^2 + (x - \delta)^2) + \gamma_t \Omega_2^2)}$$

$$\delta_1 = \delta + kv$$

$$\delta_2 = \delta - kv$$



$$\text{Integrate}\left[-\frac{\gamma_t \Omega_1 \Omega_2^2}{2 k \sqrt{\pi} v_{\text{mp}} (x + i\gamma_t + \delta) (\Gamma(\gamma_t^2 + (x - \delta)^2) + \gamma_t \Omega_2^2)}, \{x, -\text{Infinity}, \text{Infinity}\}\right]$$

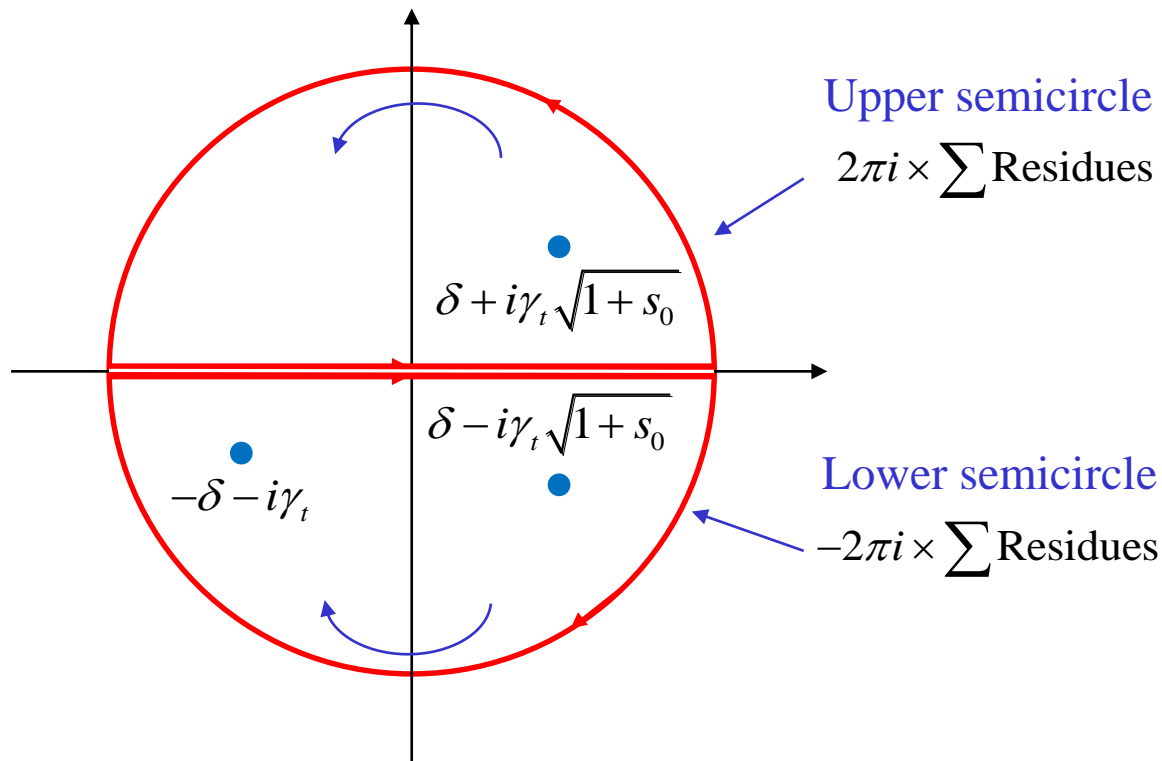
Hole-Burning Term

$$\text{Integrate}\left[-\frac{\gamma t \Omega 1 \Omega^2}{2 k \sqrt{\pi} \text{vmp}(x + i \gamma t + \delta) (\Gamma (\gamma t^2 + (x - \delta)^2) + \gamma t \Omega^2)}, \{x, -\text{Infinity}, \text{Infinity}\}\right]$$

$$\text{Solve}\left[2 k \sqrt{\pi} \text{vmp}(x + i \gamma t + \delta) (\Gamma (\gamma t^2 + (x - \delta)^2) + \gamma t \Omega^2) = 0, x\right] // \text{FullSimplify}$$

$$\left\{\{x \rightarrow -i \gamma t - \delta\}, \left\{x \rightarrow \delta - \frac{\sqrt{-\Gamma \gamma t (\Gamma \gamma t + \Omega^2)}}{\Gamma}\right\}, \left\{x \rightarrow \delta + \frac{\sqrt{-\Gamma \gamma t (\Gamma \gamma t + \Omega^2)}}{\Gamma}\right\}\right\}$$

3 poles $\rightarrow \{-i \gamma t - \delta, \delta - i \gamma t \text{Sqrt}[1 + s_0], \delta + i \gamma t \text{Sqrt}[1 + s_0]\} /. s_0 \rightarrow \frac{\Omega^2}{\Gamma \gamma t}$



Hole-Burning Term

$$\text{Residue of } \frac{g(z)}{h(z)} \text{ at } z_0 \text{ (simple pole)} = \lim_{z \rightarrow z_0} (z - z_0) \frac{g(z)}{h(z)} = \frac{g(z_0)}{h'(z_0)}$$

$$z_0 = \delta + i\gamma_t \sqrt{1 + s_0}$$

$$\text{Integrate} \left[-\frac{\gamma t \Omega_1 \Omega_2^2}{2 k \sqrt{\pi} \text{vmp} (\mathbf{x} + i \gamma t + \delta) (\Gamma (\gamma t^2 + (\mathbf{x} - \delta)^2) + \gamma t \Omega_2^2)}, \{\mathbf{x}, -\text{Infinity}, \text{Infinity}\} \right]$$

$$\text{D} \left[2 k \sqrt{\pi} \text{vmp} (\mathbf{x} + i \gamma t + \delta) (\Gamma (\gamma t^2 + (\mathbf{x} - \delta)^2) + \gamma t \Omega_2^2), \mathbf{x} \right] // \text{FullSimplify}$$

$$2 k \sqrt{\pi} \text{vmp} (\Gamma (3 x^2 + (\gamma t - i \delta)^2 + 2 i x (\gamma t + i \delta)) + \gamma t \Omega_2^2)$$

$$2 \text{ Pi I} \left(-\frac{\gamma t \Omega_1 \Omega_2^2}{2 k \sqrt{\pi} \text{vmp} (\Gamma (3 x^2 + (\gamma t - i \delta)^2 + 2 i x (\gamma t + i \delta)) + \gamma t \Omega_2^2)} \right) /. \mathbf{x} \rightarrow \delta + i \gamma t \text{Sqrt}[1 + s_0] /.$$

$$s_0 \rightarrow \frac{\Omega_2^2}{\Gamma \gamma t} // \text{FullSimplify}$$

$$\frac{i \sqrt{\pi} \Omega_1 \Omega_2^2}{2 k \text{vmp} \left(\Omega_2^2 + \Gamma \left(\gamma t + \gamma t \sqrt{1 + \frac{\Omega_2^2}{\Gamma \gamma t}} - 2 i \delta \sqrt{1 + \frac{\Omega_2^2}{\Gamma \gamma t}} \right) \right)}$$

$$\frac{i \sqrt{\pi} \Omega_1 \Omega_2^2}{2 k \text{vmp} \left(\Omega_2^2 + \Gamma \left(\gamma t + \gamma t \sqrt{1 + \frac{\Omega_2^2}{\Gamma \gamma t}} - 2 i \delta \sqrt{1 + \frac{\Omega_2^2}{\Gamma \gamma t}} \right) \right)} /. \Omega_2 \rightarrow \sqrt{Q^2 - 1} \sqrt{\Gamma} \sqrt{\gamma t} //$$

FullSimplify

$$\frac{i \sqrt{\pi} (-1 + Q^2) \gamma t \Omega_1}{2 k \text{vmp} (Q^2 \gamma t + \sqrt{Q^2} (\gamma t - 2 i \delta))}$$

$$C_0 = \frac{3\lambda^3}{8\pi^{3/2}} \frac{\Gamma N_{\text{at}}}{k v_{\text{mp}}}$$

$$Q = \sqrt{1 + s_0} \quad s_0 = \frac{\Omega_2^2}{\Gamma \gamma_t}$$

$$\chi_1 = -\frac{3\lambda^3}{4\pi^2} \cdot \frac{\Gamma N_{\text{at}}}{\Omega_1} \cdot \text{Integration} \quad \Rightarrow$$

$$\chi_1 = C_0 e^{-\delta^2 / (k v_{\text{mp}})^2} \frac{s_0 \gamma_t}{Q(2\delta + i\gamma_t(1 + Q))}$$

Coherence Term

3. Coherence term

$$\chi_2 = -\frac{3\lambda^3}{4\pi^2} \cdot \frac{\Gamma N_{\text{at}}}{\Omega_1} \cdot \frac{1}{\sqrt{\pi} v_{\text{mp}}} \int_{-\infty}^{\infty} dv e^{-(v/v_{\text{mp}})^2} \left(-\frac{\Omega_2}{(\delta - kv + i\gamma_t)} u \right)$$

$$u = \frac{(\delta_d + 2i\gamma_t) (\delta_2 - \delta_d - i\gamma_t) \Omega_1 \Omega_2}{4(\delta_2 - i\gamma_t)} \times \frac{(p - q)}{(\delta_d + i\Gamma) (\delta_2 + \delta_d + i\gamma_t) (\delta_2 - \delta_d - i\gamma_t) + (\delta_d + i\gamma_t) \Omega_2^2}$$

$$p - q = \frac{\delta_2^2 + \gamma_t^2}{\delta_2^2 + \gamma_t^2 + \frac{\gamma_t \Omega_2^2}{\Gamma}}$$

$$\frac{1}{\sqrt{\pi} v_{\text{mp}}} \star dv \star \left(\left(\Gamma (2 \, \text{i} \, \gamma t + \delta p) \left(\gamma t^2 + \delta^2 (\delta_2 - \delta p) - \text{i} \, \gamma t \, \delta p \right) \Omega_2^2 \Omega_1 \right) / \right. \\ \left. \left(4 \left(\gamma t - \text{i} (\delta_2 + \delta p) \right) \left(\Gamma \left(\gamma t^2 + \delta^2 \right) + \gamma t \, \Omega_2^2 \right) \left((\delta_2^2 + (\gamma t - \text{i} \, \delta p)^2) (\Gamma - \text{i} \, \delta p) + (\gamma t - \text{i} \, \delta p) \, \Omega_2^2 \right) \right) / \right. \\ \left. \delta p \rightarrow (2 \, x) / . \, \delta_1 \rightarrow (\delta + x) / . \, \delta_2 \rightarrow (\delta - x) / . \, v \rightarrow (x / k) / . \, dv \rightarrow (dx / k) // \text{FullSimplify} \right) \\ \frac{dx \, \Gamma (x + \text{i} \, \gamma t) \left(3 x^2 - 2 \, \text{i} \, x \, \gamma t + \gamma t^2 - 4 x \, \delta + \delta^2 \right) \Omega_1 \Omega_2^2}{2 k \sqrt{\pi} v_{\text{mp}} (\gamma t - \text{i} (x + \delta)) \left(\Gamma \left(\gamma t^2 + (x - \delta)^2 \right) + \gamma t \, \Omega_2^2 \right) \left((-2 \, \text{i} \, x + \Gamma) \left((-2 \, \text{i} \, x + \gamma t)^2 + (x - \delta)^2 \right) + (-2 \, \text{i} \, x + \gamma t) \, \Omega_2^2 \right)}$$

$$\text{Integrate} \left[\frac{\Gamma (x + \text{i} \, \gamma t) \left(3 x^2 - 2 \, \text{i} \, x \, \gamma t + \gamma t^2 - 4 x \, \delta + \delta^2 \right) \Omega_1 \Omega_2^2}{2 k \sqrt{\pi} v_{\text{mp}} (\gamma t - \text{i} (x + \delta)) \left(\Gamma \left(\gamma t^2 + (x - \delta)^2 \right) + \gamma t \, \Omega_2^2 \right) \left((-2 \, \text{i} \, x + \Gamma) \left((-2 \, \text{i} \, x + \gamma t)^2 + (x - \delta)^2 \right) + (-2 \, \text{i} \, x + \gamma t) \, \Omega_2^2 \right)}, \{x, -\text{Infinity}, \text{Infinity}\} \right]$$

Coherence Term

$$\text{Solve}\left[2 k \sqrt{\pi} u (\gamma t - i (x + \delta)) \left(\Gamma (\gamma t^2 + (x - \delta)^2) + \gamma t \Omega^2\right) = 0, x\right] // \text{FullSimplify} // \text{Expand}$$

$$\left\{\{x \rightarrow -i \gamma t - \delta\}, \left\{x \rightarrow \delta - \frac{\sqrt{-\Gamma \gamma t (\Gamma \gamma t + \Omega^2)}}{\Gamma}\right\}, \left\{x \rightarrow \delta + \frac{\sqrt{-\Gamma \gamma t (\Gamma \gamma t + \Omega^2)}}{\Gamma}\right\}\right\}$$

$$\text{NSolve}\left[\left((-2 i x + \Gamma) \left((-2 i x + \gamma t)^2 + (x - \delta)^2\right) + (-2 i x + \gamma t) \Omega^2\right) = 0, x\right] /. \Gamma \rightarrow 1 /. \gamma t \rightarrow 0.5 /. \Omega^2 \rightarrow 12.3 /. \delta \rightarrow 9.235 /. k \rightarrow 1 /. u \rightarrow 1 /. \Omega 1 \rightarrow 2.5$$

$$\{\{x \rightarrow -12.4769 - 0.496267 i\}, \{x \rightarrow 0.00424836 - 0.340128 i\}, \{x \rightarrow 6.31598 - 0.330271 i\}\}$$

6 poles: 1 upper half plane $\rightarrow z_0 = \delta + i \gamma_t \sqrt{1 + s_0}$

5 lower half plane

$$\text{D}\left[2 k \sqrt{\pi} u (\gamma t - i (x + \delta)) \left(\Gamma (\gamma t^2 + (x - \delta)^2) + \gamma t \Omega^2\right) \left((-2 i x + \Gamma) \left((-2 i x + \gamma t)^2 + (x - \delta)^2\right) + (-2 i x + \gamma t) \Omega^2\right), x\right] // \text{FullSimplify}$$

$$2 k \sqrt{\pi} u \left(2 (x + i \gamma t + \delta) \left(9 x^2 - 2 \Gamma \gamma t - \gamma t^2 + i \Gamma \delta - \delta^2 + x (3 i \Gamma + 8 i \gamma t + 4 \delta) - \Omega^2\right) \left(\Gamma (\gamma t^2 + (x - \delta)^2) + \gamma t \Omega^2\right) + 2 \Gamma (x - \delta) (\gamma t - i (x + \delta)) \left((-2 i x + \Gamma) \left((-2 i x + \gamma t)^2 + (x - \delta)^2\right) + (-2 i x + \gamma t) \Omega^2\right) - i \left(\Gamma (\gamma t^2 + (x - \delta)^2) + \gamma t \Omega^2\right) \left((-2 i x + \Gamma) \left((-2 i x + \gamma t)^2 + (x - \delta)^2\right) + (-2 i x + \gamma t) \Omega^2\right)\right)$$

$$2 \text{ Pi } I \left(\left(\Gamma (x + i \gamma t) \left(3 x^2 - 2 i x \gamma t + \gamma t^2 - 4 x \delta + \delta^2\right) \Omega 1 \Omega^2 \right) / \left(2 k \sqrt{\pi} u \left(2 (x + i \gamma t + \delta) \left(9 x^2 - 2 \Gamma \gamma t - \gamma t^2 + i \Gamma \delta - \delta^2 + x (3 i \Gamma + 8 i \gamma t + 4 \delta) - \Omega^2 \right) \left(\Gamma (\gamma t^2 + (x - \delta)^2) + \gamma t \Omega^2 \right) + 2 \Gamma (x - \delta) (\gamma t - i (x + \delta)) \left((-2 i x + \Gamma) \left((-2 i x + \gamma t)^2 + (x - \delta)^2 \right) + (-2 i x + \gamma t) \Omega^2 \right) - i \left(\Gamma (\gamma t^2 + (x - \delta)^2) + \gamma t \Omega^2 \right) \left((-2 i x + \Gamma) \left((-2 i x + \gamma t)^2 + (x - \delta)^2 \right) + (-2 i x + \gamma t) \Omega^2 \right) \right) \right) / . x \rightarrow \delta + I \gamma t \text{ Sqrt}[1 + s_0] /. s_0 \rightarrow \frac{\Omega^2}{\Gamma \gamma t} // \text{FullSimplify}$$

Coherence Term

$$\begin{aligned}\chi_2 &= iC_0 e^{-\delta^2/(kv_{\text{mp}})^2} \cdot \frac{s_0(1-Q)\Gamma\gamma_t}{2Q} \\ &\quad \times \frac{(\delta + i(1+Q)\gamma_t)(2\delta + i(1+3Q)\gamma_t)}{(\delta + iQ\gamma_t)(2\delta + i(1+Q)\gamma_t)Z}, \\ Z &= (2\delta + i(1+Q)\gamma_t) \cdot (2\delta + i(1+3Q)\gamma_t) \\ &\quad + i\Gamma(2\delta + i(1+Q)^2\gamma_t),\end{aligned}$$

$$Q = \sqrt{1+s_0}$$

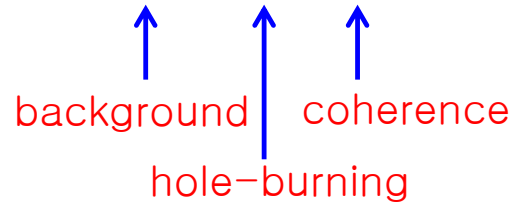
$$s_0 = \frac{\Omega_2^2}{\Gamma\gamma_t}$$

$$C_0 = \frac{3\lambda^3}{8\pi^{3/2}} \frac{\Gamma N_{\text{at}}}{kv_{\text{mp}}}$$

Summary for Susceptibility

Doppler-
averaged
Susceptibility

$$\chi = \chi_0 + \chi_1 + \chi_2$$



background hole-burning coherence

$$\begin{aligned}\chi_0 &= C_0 e^{-\delta^2/(kv_{\text{mp}})^2} \left[-\text{Erfi}\left(\frac{\delta}{kv_{\text{mp}}}\right) + i \right], \\ \chi_1 &= C_0 e^{-\delta^2/(kv_{\text{mp}})^2} \frac{s_0 \gamma_t}{Q(2\delta + i\gamma_t(1+Q))}, \\ \chi_2 &= iC_0 e^{-\delta^2/(kv_{\text{mp}})^2} \cdot \frac{s_0(1-Q)\Gamma\gamma_t}{2Q} \\ &\quad \times \frac{(\delta + i(1+Q)\gamma_t)(2\delta + i(1+3Q)\gamma_t)}{(\delta + iQ\gamma_t)(2\delta + i(1+Q)\gamma_t)Z}, \\ Z &= (2\delta + i(1+Q)\gamma_t) \cdot (2\delta + i(1+3Q)\gamma_t) \\ &\quad + i\Gamma(2\delta + i(1+Q)^2\gamma_t),\end{aligned}$$

where

$$Q = \sqrt{1 + s_0}, \quad s_0 = \frac{\Omega_2^2}{\Gamma\gamma_t}$$

$$C_0 = \frac{3\lambda^3}{8\pi^{3/2}} \frac{\Gamma N_{\text{at}}}{kv_{\text{mp}}}$$

$$\begin{aligned}\chi &= C_0 e^{-\delta^2/(kv_{\text{mp}})^2} \left[-\text{Erfi}\left(\frac{\delta}{kv_{\text{mp}}}\right) + i \right] \\ &\quad + C_0 e^{-\delta^2/(kv_{\text{mp}})^2} \cdot \frac{s_0 \gamma_t}{Q} \cdot \frac{Y}{Z},\end{aligned}$$

$$Y = 2\delta + i(1+3Q)\gamma_t + \frac{i}{2}(3-Q)\Gamma - \frac{(1+Q)\Gamma\gamma_t}{2(\delta + iQ\gamma_t)}$$

Absorption coefficient

Total

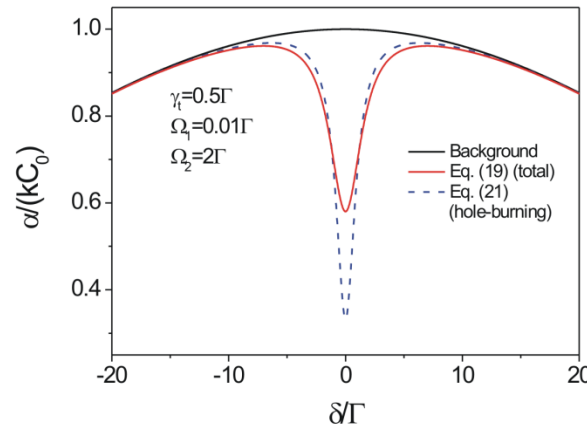
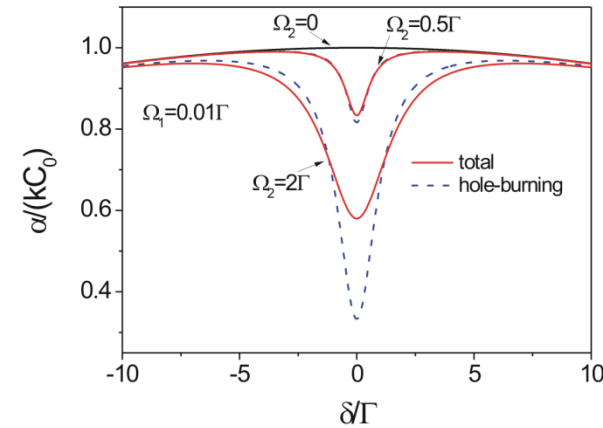
$$\begin{aligned}\alpha &= kC_0 e^{-(\delta/kv_{\text{mp}})^2} (1 - f(\delta)), \\ f(\delta) &= -\text{Im} \left[\frac{s_0 \gamma_t}{Q} \cdot \frac{Y}{Z} \right].\end{aligned}$$

Hole-burning

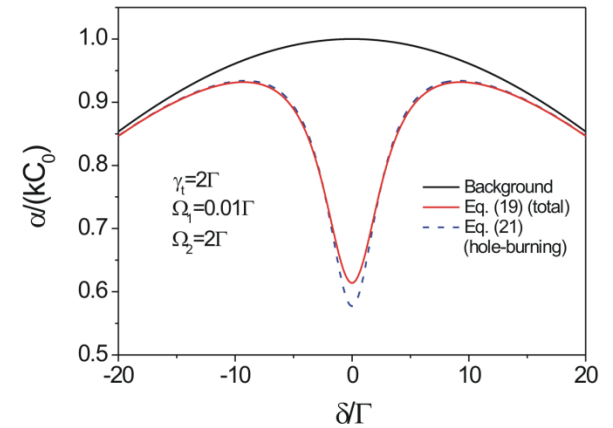
$$\begin{aligned}\alpha_1 &= kC_0 e^{-(\delta/kv_{\text{mp}})^2} (1 - f_1(\delta)), \\ f_1(\delta) &= \frac{s_0 \gamma_t^2 (1+Q)}{Q(4\delta^2 + (1+Q)^2 \gamma_t^2)},\end{aligned}$$

Result 1

Dependence of Pump
beam Rabi frequency



γ_t dependence



(a)

(b)

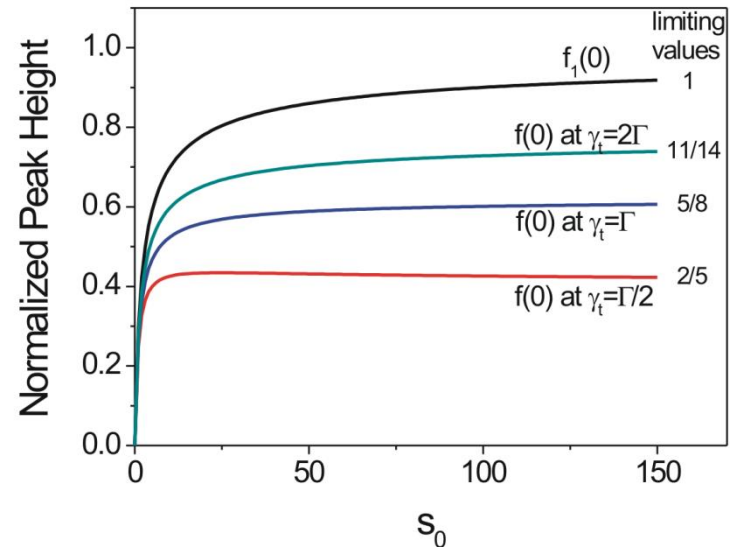
Peak Height

$$f(0) = \frac{s_0[(1 + Q(4 - Q))\Gamma + 2Q(1 + 3Q)\gamma_t]}{2Q^2(1 + Q)[(1 + Q)\Gamma + (1 + 3Q)\gamma_t]},$$

$$f_1(0) = \frac{s_0}{Q(1 + Q)},$$

$f(0)$ at $s_0 \rightarrow \infty$

$$\frac{6\gamma_t - \Gamma}{2(3\gamma_t + \Gamma)}$$



Result 2

Linewidths

Total

Hole-burning

$$2Q(1+Q)\gamma_t[(1+Q)\Gamma + (1+3Q)\gamma_t] \\ \times [(1+Q(4-Q))\Gamma + 2Q(1+3Q)\gamma_t] \\ \times [4Q^2(1+3Q)^2\gamma_t^2 + (1+5Q+Q^2(11-Q))\Gamma^2 \\ + (1+3Q)(1+6Q+Q^2(13-4Q))\Gamma\gamma_t]^{-1}.$$

$$(1+Q)\gamma_t = (1+\sqrt{1+s_0})\gamma_t$$

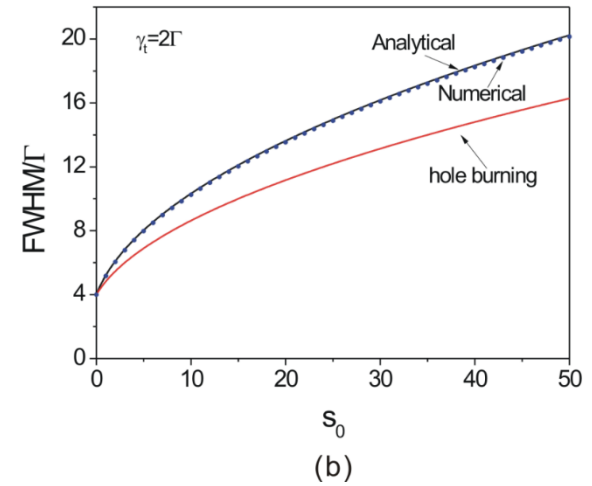
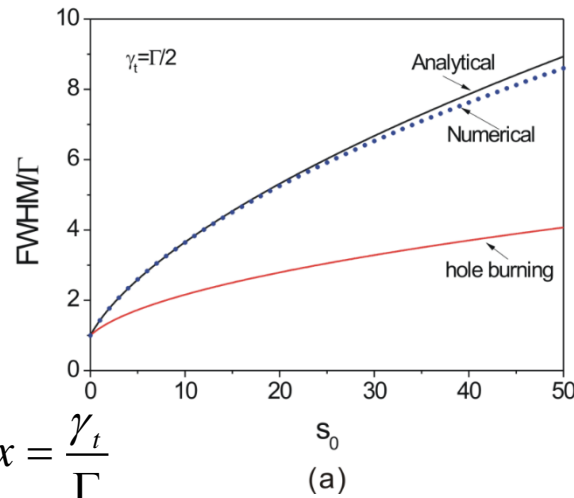
Empirical formula

$$(1 + (1 + as_0)^b)\gamma_t$$

$$a = 1 + \frac{a_1 + a_2x}{a_3 + a_4x + x^2}$$

$$b = \frac{1}{2} + \frac{b_1 + b_2x}{b_3 + b_4x + x^2}$$

$$x = \frac{\gamma_t}{\Gamma}$$



$$a_1 = 0.265, a_2 = 1.14, a_3 = 0.210, a_4 = 0.0480$$

$$b_1 = 0.0326, b_2 = -0.0109, b_3 = 0.0548, b_4 = 0.221$$

G. Moon and H. R. Noh, Appl. Opt. **57**, 3881 (2018).

Contents

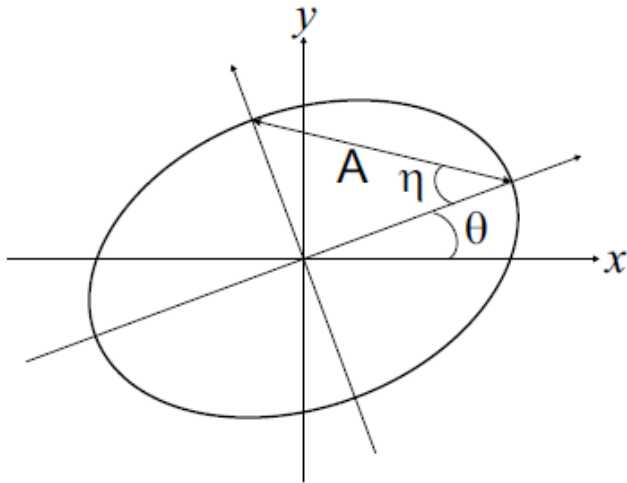
- **Two-Level Atoms**

- One field (Optical Bloch equations)
- One field (Linear absorption)
- One field (Analytical solutions to Optical Bloch equations)
- Two fields (Susceptibility)
- **Two fields (Polarization spectroscopy for the $F_g=0 \rightarrow F_e=1$ transition)**

- **Multi-Level Atoms**

- Density matrix equations
- Electromagnetically induced transparency and absorption
- Wave mixing and rate equations
- Elliptic polarization and arbitrary quantization axis

Elliptic Polarization



$$\vec{E} = Ae^{i\phi} \left[-e^{-i\theta} \sin\left(\eta + \frac{\pi}{4}\right) \hat{e}_+ - e^{i\theta} \sin\left(\eta - \frac{\pi}{4}\right) \hat{e}_- \right] e^{i(kz - \omega t)},$$

**In an circular
anisotropic medium**

**Traversing a cell of
length z**

$$A = \frac{A_0}{\sqrt{2}} [a_+^2 + a_-^2 + (a_+^2 - a_-^2) \sin 2\eta_0]^{1/2},$$

$$\phi = \frac{k}{4} (\chi_+^r + \chi_-^r) z,$$

$$\theta = \theta_0 + \frac{k}{4} (\chi_-^r - \chi_+^r) z,$$

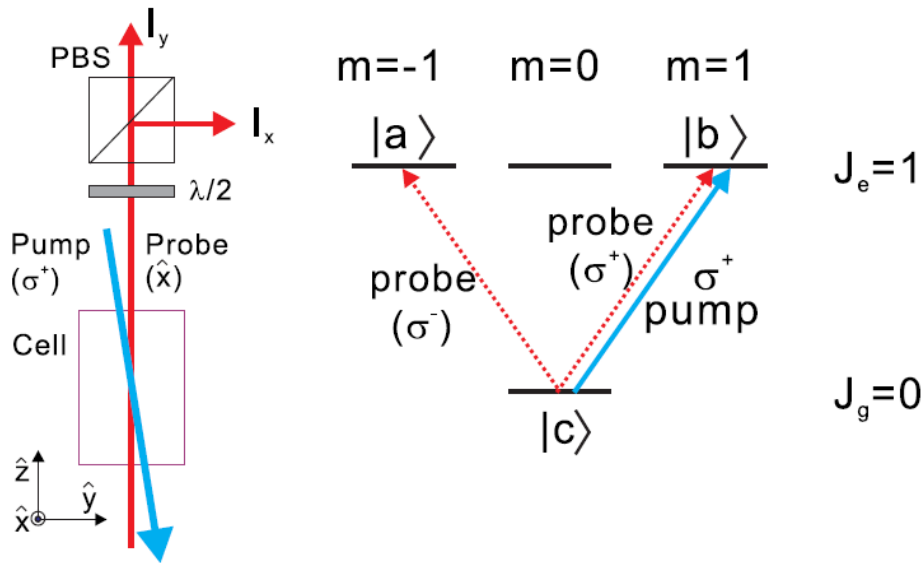
$$\eta = \tan^{-1} \left[\frac{a_+ - a_- + (a_+ + a_-) \tan \eta_0}{a_+ + a_- + (a_+ - a_-) \tan \eta_0} \right],$$

where

$$a_{\pm} = \exp(-k\chi_{\pm}^i z/2).$$

M. J. Seo et al. JKPS **59**, 253 (2011).

Polarization Spectroscopy



The signal of PS:
the difference in the intensities of
the two orthogonal linear
components of the probe beam
along the x and y axes.

$$\Delta I = I_0 a_+ a_- \sin 2\theta$$

$$\theta = \frac{kL}{4} \Delta\chi_r \quad \leftarrow \text{Polarization rotation angle}$$

$$a_{\pm} = \exp\left(-\frac{kL}{2} \text{Im}(\chi_{\pm})\right)$$

$$\Delta I = I_x - I_y$$

$$\Delta\chi_r = \text{Re}(\chi_- - \chi_+)$$

$$\rho_{ac} = z_1 e^{-i\delta_d t},$$

$$\rho_{bc} = z_2 + z_3 e^{-i\delta_d t} + z_4 e^{i\delta_d t},$$

$$\rho_{ab} = z_5 e^{-i\delta_d t},$$

$$\rho_{aa} = p_1 + u_1 e^{-i\delta_d t} + u_1^* e^{i\delta_d t},$$

$$\rho_{bb} = p_2 + u_2 e^{-i\delta_d t} + u_2^* e^{i\delta_d t},$$

$$\rho_{cc} = p_3 + u_3 e^{-i\delta_d t} + u_3^* e^{i\delta_d t}.$$

$$p_1 + p_2 + p_3 = 1$$

$$u_1 + u_2 + u_3 = 0$$

Polarization Spectroscopy

$$\chi_- = C_0 e^{-\delta^2/(kv_{\text{mp}})^2} \left[-\text{Erfi}\left(\frac{\delta}{kv_{\text{mp}}}\right) + i \right]$$

$$+ C_0 e^{-\delta^2/(kv_{\text{mp}})^2} \frac{s_0 \gamma_t}{Q} \frac{1}{X},$$

$$\chi_+ = C_0 e^{-\delta^2/(kv_{\text{mp}})^2} \left[-\text{Erfi}\left(\frac{\delta}{kv_{\text{mp}}}\right) + i \right]$$

$$+ C_0 e^{-\delta^2/(kv_{\text{mp}})^2} \frac{s_0 \gamma_t}{Q} \frac{Y}{Z},$$

$$Q = \sqrt{1 + s_0}$$

$$s_0 = \frac{\Omega_2^2}{\Gamma \gamma_t}$$

$$C_0 = \frac{3\lambda^3}{8\pi^{3/2}} \frac{\Gamma N_{\text{at}}}{kv_{\text{mp}}}$$

$$X = 4\delta + i[2(1 + Q)\gamma_t + (Q - 1)\Gamma]$$

$$- \frac{(Q - 1)(Q - 3)\Gamma(\Gamma - \gamma_t)}{4\delta + i[(3 - Q)\Gamma + 4Q\gamma_t]},$$

$$Y = 2\delta + i(1 + 3Q)\gamma_t + \frac{i}{2}(3 - Q)\Gamma - \frac{(1 + Q)\Gamma\gamma_t}{2(\delta + iQ\gamma_t)}$$

$$Z = (2\delta + i(1 + Q)\gamma_t)(2\delta + i(1 + 3Q)\gamma_t) \\ + i\Gamma(2\delta + i(1 + Q)^2\gamma_t).$$

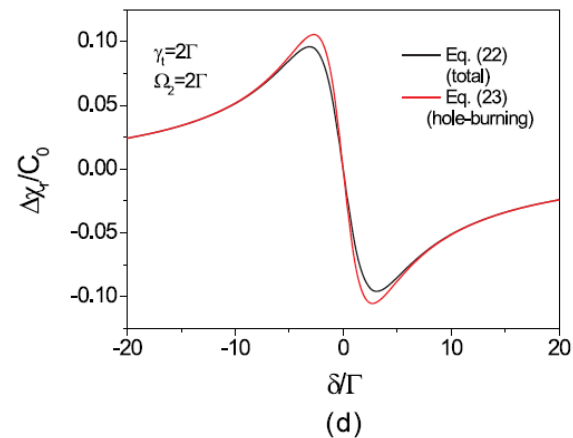
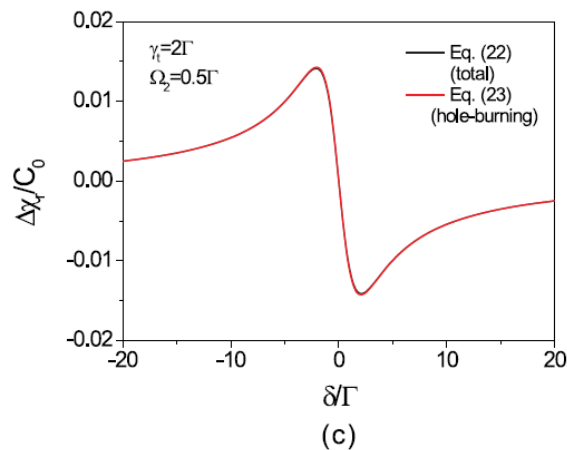
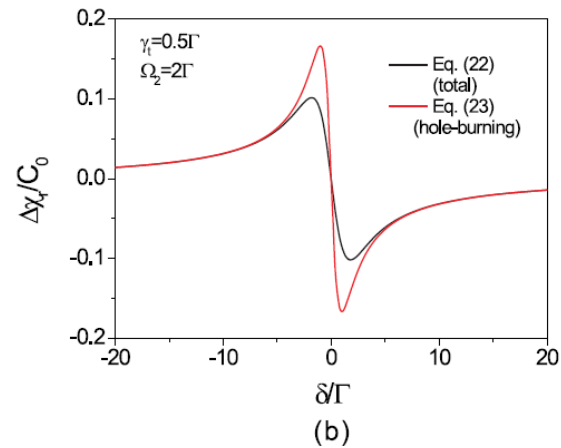
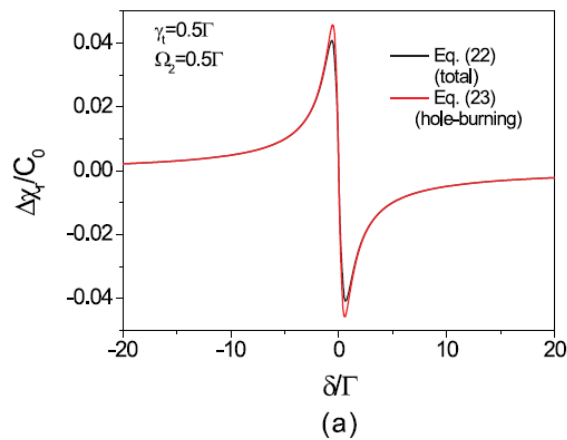
Polarization Spectroscopy

The PS signal

$$\Delta\chi_r = \frac{C_0 s_0 \gamma_t}{Q} \operatorname{Re} \left(\frac{1}{X} - \frac{Y}{Z} \right) e^{-\delta^2 / (k v_{\text{mp}})^2}$$

The hole-burning term

$$\Delta\chi_r^{\text{h.b.}} = -C_0 \frac{s_0 \gamma_t}{Q} \frac{\delta}{(4\delta^2 + (1 + Q)^2 \gamma_t^2)} e^{-\delta^2 / (k v_{\text{mp}})^2}$$



Contents

- **Two-Level Atoms**

- One field (Optical Bloch equations)
- One field (Linear absorption)
- One field (Analytical solutions to Optical Bloch equations)
- Two fields (Susceptibility)
- Two fields (Polarization spectroscopy for the $F_g=0 \rightarrow F_e=1$ transition)

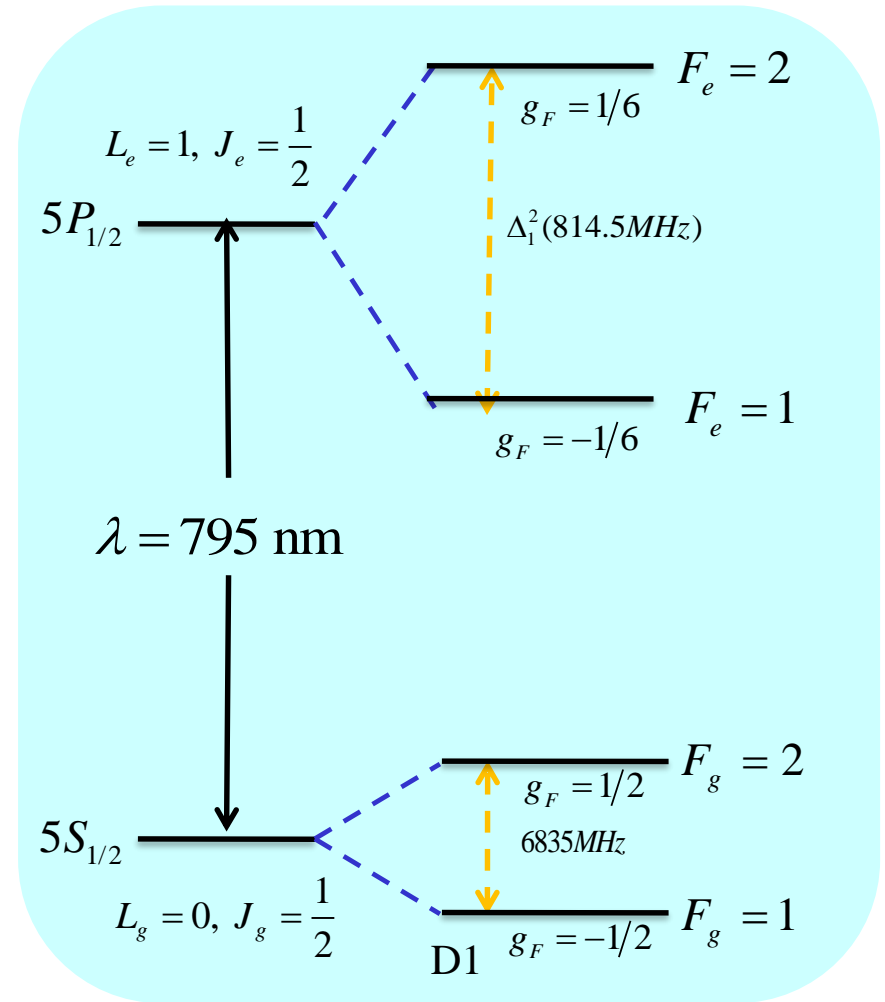
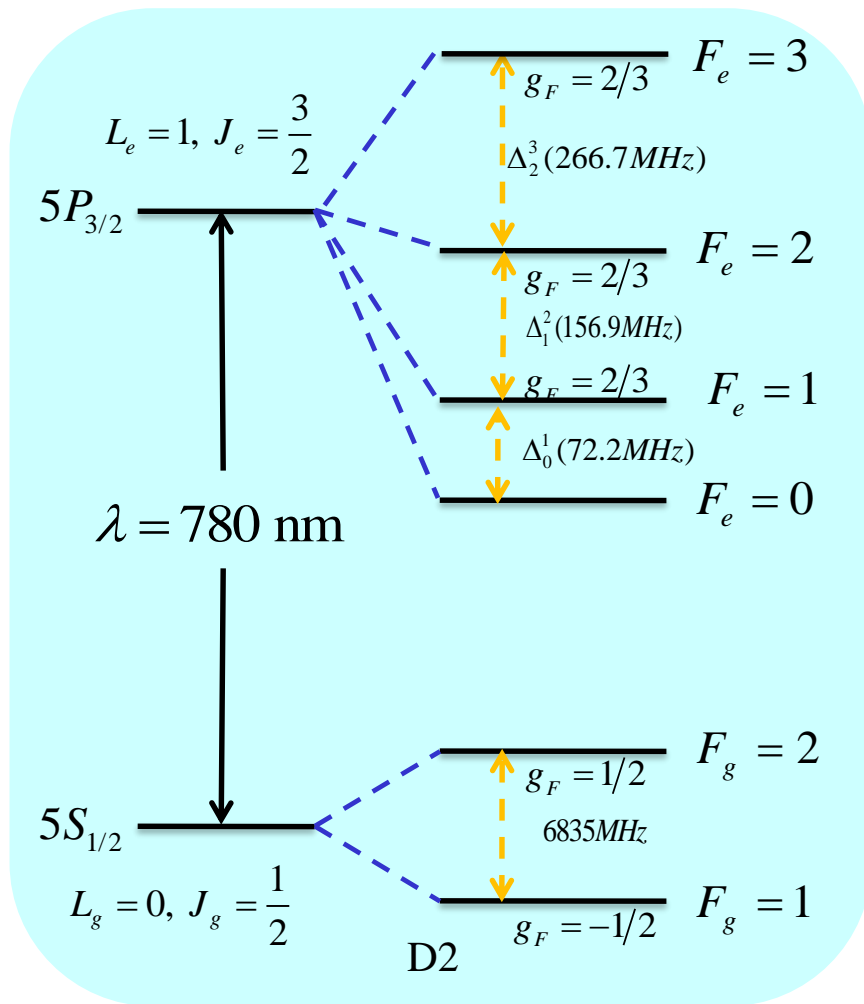
- **Multi-Level Atoms**

- **Density matrix equations**

- Electromagnetically induced transparency and absorption
 - Wave mixing and rate equations
 - Elliptic polarization and arbitrary quantization axis

Multilevel Atoms

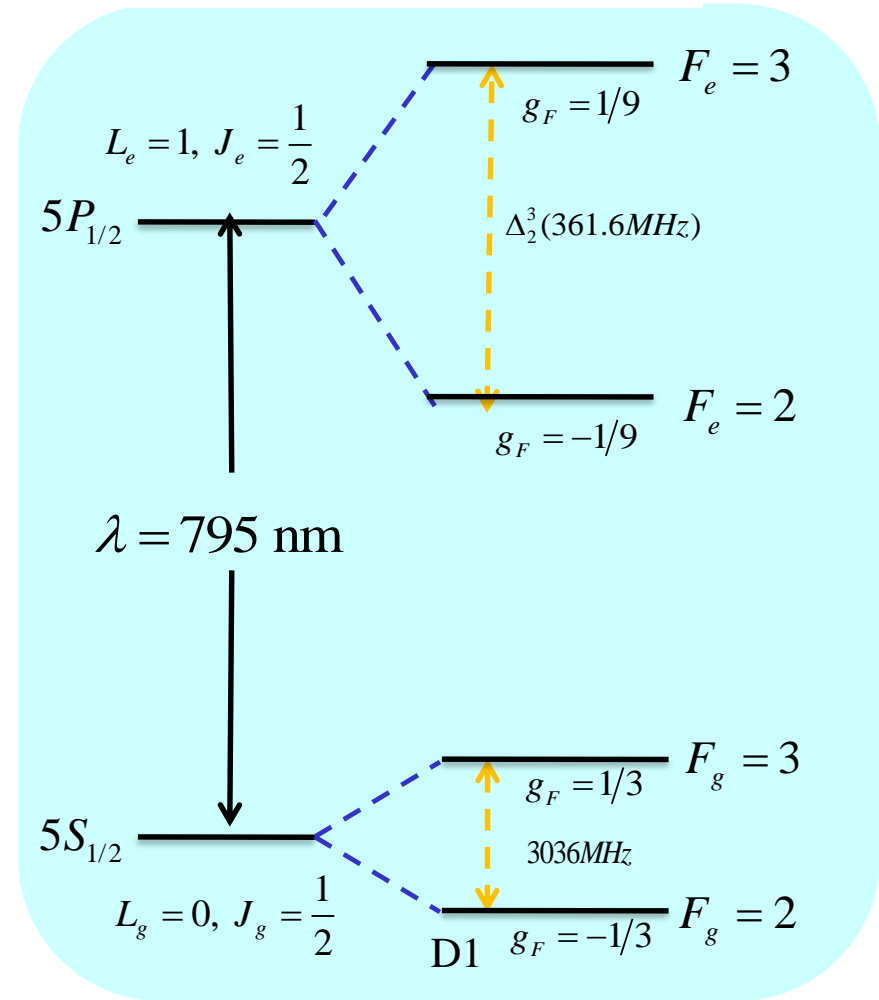
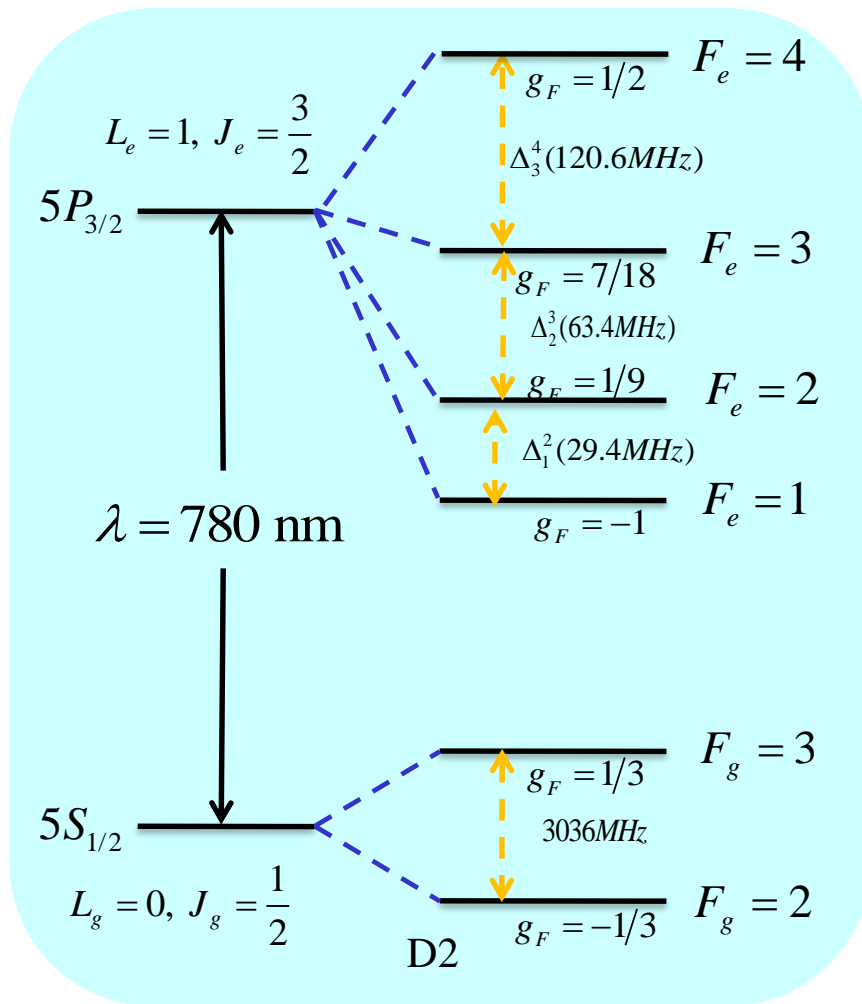
Energy level diagram of the ^{87}Rb atom



$$I = \frac{3}{2}, \quad S = \frac{1}{2}$$

Density Matrix Equations

Energy level diagram of the ^{85}Rb atom



$$I = \frac{5}{2}, \quad S = \frac{1}{2}$$

Density Matrix Equations

1. Magnetic field
2. Arbitrary polarization
3. Single field
4. $F_g = 2 \rightarrow F_e = 3$

$$\dot{\rho} = -\frac{i}{\hbar}[H_A + V, \rho] + (\dot{\rho})_{\text{sp}}$$

$$H_A = \sum_{m=-3}^3 (\hbar\omega_0 + \mu_B g_e B m_e) |e_m\rangle\langle e_m| + \sum_{m=-2}^2 \mu_B g_g B m_g |g_m\rangle\langle g_m|$$

$$V = \frac{1}{2} \hbar \Omega_1 \sum_{q=-1}^1 \sum_{m=-2}^2 c_q C_{F_g=2, m}^{F_e=3, m+q} |e_{m+q}\rangle\langle g_m| e^{-i\omega_1 t} + c.c$$

$$5P_{3/2} \begin{array}{ccccccc} & -3 & -2 & -1 & 0 & 1 & 2 & 3 \\ \hline & \text{---} & \text{---} & \text{---} & \text{---} & \text{---} & \text{---} & \text{---} \end{array} F_e = 3$$

$$5S_{1/2} \begin{array}{ccccccc} & & & & & & & \\ \hline & & \text{---} & \text{---} & \text{---} & \text{---} & \text{---} & \\ & -2 & -1 & 0 & 1 & 2 & & \end{array} F_g = 2$$

$$\vec{E} = \hat{\varepsilon} \frac{E}{2} e^{-i\omega_L t} + c.c$$

$$\hat{\varepsilon} = c_+ \hat{\varepsilon}_+ + c_- \hat{\varepsilon}_- + c_0 \hat{\varepsilon}_0$$

$$\hat{\varepsilon}_- = \frac{1}{\sqrt{2}} (\hat{x} - i\hat{y})$$

$$\hat{\varepsilon}_+ = -\frac{1}{\sqrt{2}} (\hat{x} + i\hat{y})$$

$$\hat{\varepsilon}_0 = \hat{z}$$

Dipole Moment

$$\langle F_e, m_{F_e} | -\vec{d} \cdot E_0 \hat{\epsilon}_q | F_g, m_{F_g} \rangle = \hbar \Omega_1 C_{F_g, m_{F_g}}^{F_g, m_{F_g}}$$

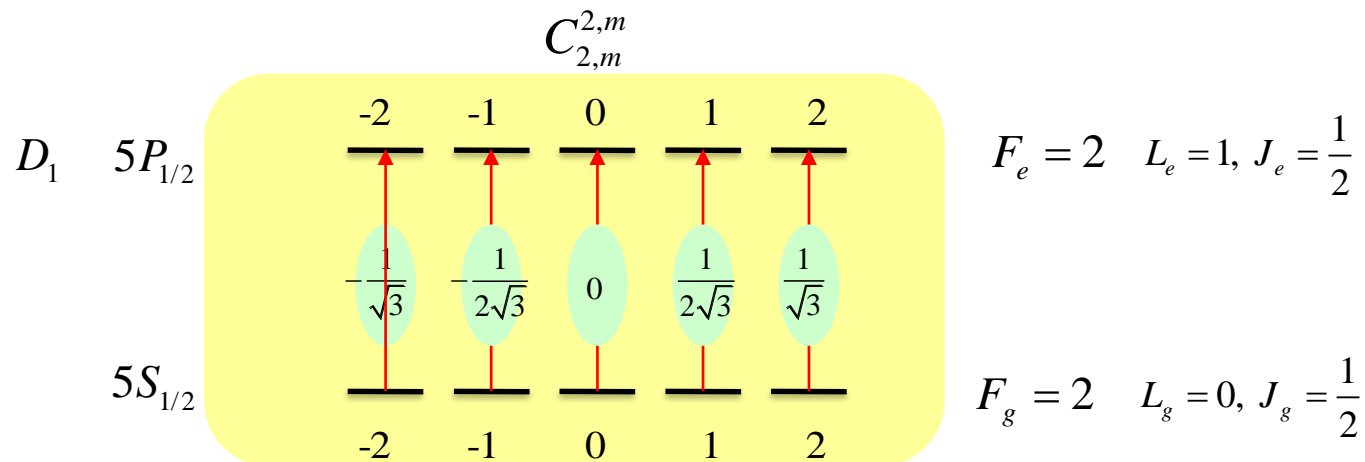
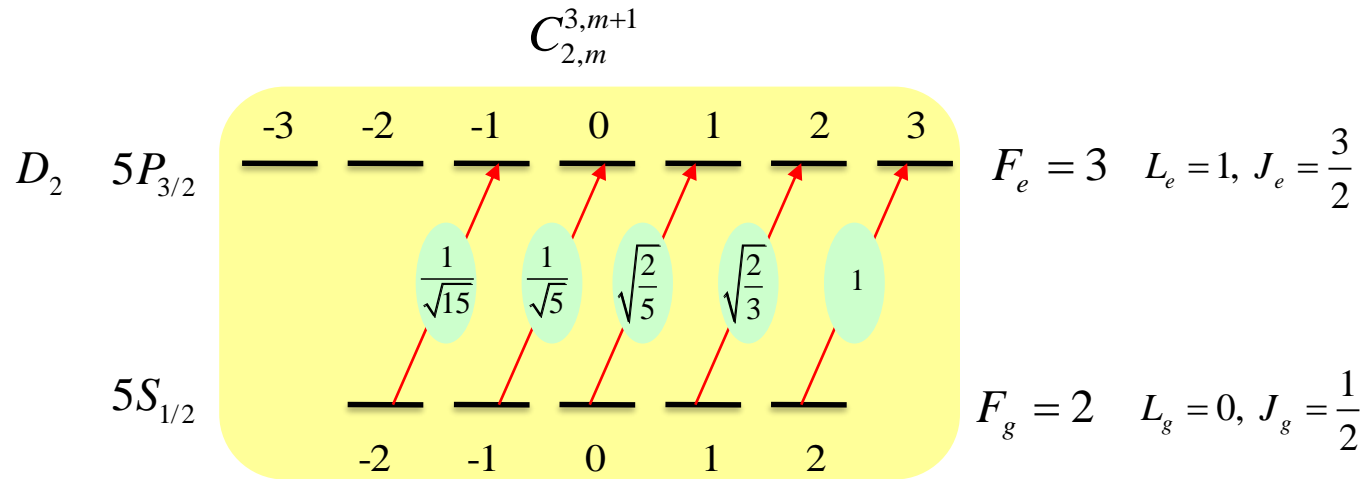
$$C_{F_g, m_{F_g}}^{F_g, m_{F_g}} = (-1)^{2F_e + I + 2J_g + L_e + S - m_{F_e}} \\ \times \sqrt{(2L_e + 1)(2F_g + 1)(2F_e + 1)(2J_g + 1)(2J_e + 1)} \\ \times \begin{Bmatrix} L_e & 1 & L_g \\ J_g & S & J_e \end{Bmatrix} \begin{Bmatrix} J_e & 1 & J_g \\ F_g & I & F_e \end{Bmatrix} \begin{pmatrix} F_e & 1 & F_g \\ -m_{F_e} & q & m_{F_g} \end{pmatrix}$$

$$\hbar \Omega_1 = -d_{ge} E_0$$

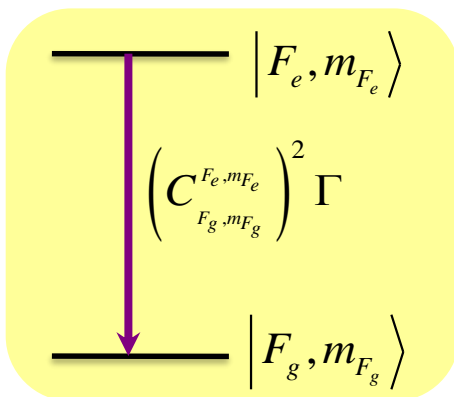
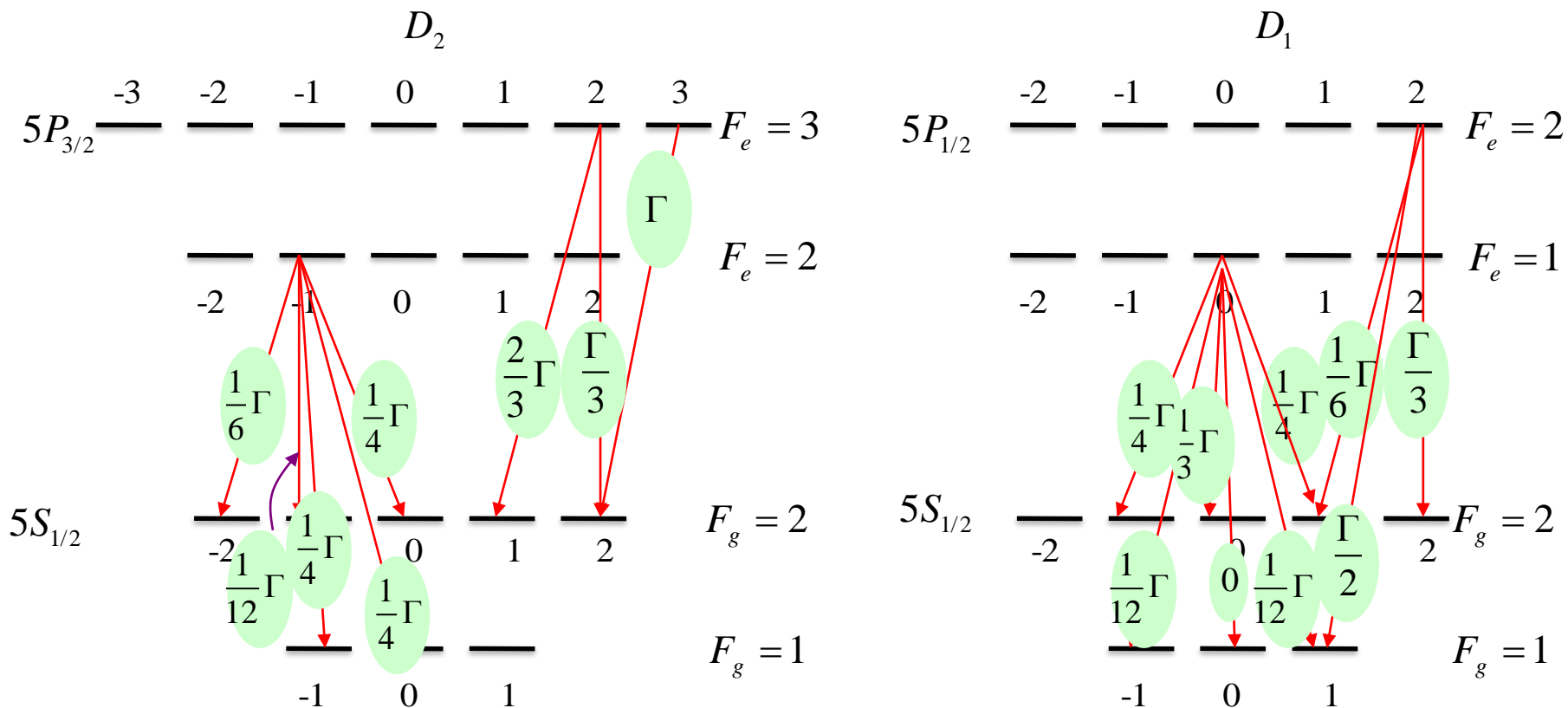
$$\Gamma = \frac{1}{3\pi\epsilon_0} \frac{\omega_0^3}{\hbar c^3} d_{eg}^2$$

	λ	Γ
D2	780.241 nm	$2\pi \times 6.065$ MHz
D1	794.979 nm	$2\pi \times 5.746$ MHz

Examples



Decay Rates



$$C_{F_g, m_{F_g}}^{F_e, m_{F_e}} = (-1)^{2F_e + I + 2J_g + L_e + S - m_{F_e}} \times \sqrt{(2L_e + 1)(2F_g + 1)(2F_e + 1)(2J_g + 1)(2J_e + 1)} \times \begin{Bmatrix} L_e & 1 & L_g \\ J_g & S & J_e \end{Bmatrix} \begin{Bmatrix} J_e & 1 & J_g \\ F_g & I & F_e \end{Bmatrix} \begin{pmatrix} F_e & 1 & F_g \\ -m_{F_e} & q & m_{F_g} \end{pmatrix}$$

Atomic Hamiltonian

1. $B=0$
2. σ^+ polarization
3. $F_g = 2 \rightarrow F_e = 3$

12X12 Matrices

The diagram shows a Hamiltonian matrix $H_A = \hbar$ acting on a basis of states $e_{-3}, e_{-2}, e_{-1}, e_0, e_1, e_2, e_3, g_{-2}, g_{-1}, g_0, g_1, g_2$. The matrix is block-diagonal, with a 6x6 block of ω_0 and a 5x5 block of 0s.

Interaction Hamiltonian

$$V = \frac{1}{2} \hbar \Omega_1 \begin{pmatrix} & & & & 0 & 0 & \frac{1}{\sqrt{15}} e^{-i\omega t} & & 0 \\ & & & & 0 & & \frac{1}{\sqrt{5}} e^{-i\omega t} & & 0 \\ & & 0 & & 0 & & \sqrt{\frac{2}{5}} e^{-i\omega t} & & \\ & & & & 0 & & \sqrt{\frac{2}{3}} e^{-i\omega t} & & \\ & & & & & & e^{-i\omega t} & & \\ 0 & 0 & \frac{1}{\sqrt{15}} e^{i\omega t} & & & & & & \\ & \frac{1}{\sqrt{5}} e^{i\omega t} & & 0 & & & & & \\ & \sqrt{\frac{2}{5}} e^{i\omega t} & & & & & & & \\ & \sqrt{\frac{2}{3}} e^{i\omega t} & & & & & & & \\ & e^{i\omega t} & & & & & & & 0 \end{pmatrix}$$

Spontaneous Emissions

$$\langle e_m | (\dot{\rho}_{\text{sp}}) | e_{m'} \rangle = -\Gamma \langle e_m | \rho | e_{m'} \rangle$$

$$\langle e_m | (\dot{\rho}_{\text{sp}}) | g_{m'} \rangle = -\frac{\Gamma}{2} \langle e_m | \rho | g_{m'} \rangle$$

$$\langle g_m | (\dot{\rho}_{\text{sp}}) | e_{m'} \rangle = -\frac{\Gamma}{2} \langle g_m | \rho | e_{m'} \rangle$$

$$\langle g_m | (\dot{\rho}_{\text{sp}}) | g_{m'} \rangle = \Gamma \sum_{q=-1}^1 C_{F_g=2,m}^{F_e=3,m+q} C_{F_g=2,m'}^{F_e=3,m'+q} \langle g_m | \rho | g_{m'} \rangle$$

$$\begin{array}{ccccccc}
 & & & D_2 & & & \\
 5P_{3/2} & \overline{e_{-3}} & \overline{e_{-2}} & \overline{e_{-1}} & \overline{e_0} & \overline{e_1} & \overline{e_2} & \overline{e_3} & F_e = 3 \\
 \\
 5S_{1/2} & & \overline{g_{-2}} & \overline{g_{-1}} & \overline{g_0} & \overline{g_1} & \overline{g_2} & & F_g = 2
 \end{array}$$

Examples $\langle g_2 | (\dot{\rho}_{\text{sp}}) | g_2 \rangle = \Gamma \left(\frac{1}{15} \rho_{e_1, e_1} + \frac{1}{3} \rho_{e_2, e_2} + \rho_{e_3, e_3} \right)$

$$\langle g_1 | (\dot{\rho}_{\text{sp}}) | g_2 \rangle = \Gamma \left(\sqrt{\frac{2}{3}} \rho_{e_0, e_1} + \frac{2}{3} \sqrt{\frac{2}{5}} \rho_{e_1, e_2} + \frac{1}{5\sqrt{3}} \rho_{e_2, e_3} \right)$$

Spontaneous Emissions

$$\dot{\rho}_{\text{sp}} = \left(\begin{array}{c|c} -\Gamma \rho_{ij} & -\frac{\Gamma}{2} \rho_{ij} \\ \hline -\frac{\Gamma}{2} \rho_{ij} & 5 \times 5 \end{array} \right)$$

$$5P_{3/2} \xrightarrow{e_{-3}} \xrightarrow{e_{-2}} \xrightarrow{e_{-1}} \xrightarrow{D_2} \xrightarrow{e_0} \xrightarrow{e_1} \xrightarrow{e_2} \xrightarrow{e_3} F_e = 3$$

$$5S_{1/2} \quad \xrightarrow{g_{-2}} \xrightarrow{g_{-1}} \xrightarrow{g_0} \xrightarrow{g_1} \xrightarrow{g_2} \quad F_g = 2$$

$$\langle g_m | (\dot{\rho}_{\text{sp}}) | g_{m'} \rangle = \Gamma \sum_{q=-1}^1 C_{F_g=2,m}^{F_e=3,m+q} C_{F_g=2,m'}^{F_e=3,m'+q} \langle g_m | \rho | g_{m'} \rangle$$

$$5 \times 5 = \Gamma \left(\begin{array}{ccc} \rho_{1,1} + \frac{1}{3} \rho_{2,2} + \frac{1}{15} \rho_{3,3} & \sqrt{\frac{2}{3}} \rho_{1,2} + \frac{2}{3} \sqrt{\frac{2}{5}} \rho_{2,3} + \frac{1}{5\sqrt{3}} \rho_{3,4} & \cdot \quad \cdot \quad \cdot \\ \sqrt{\frac{2}{3}} \rho_{2,1} + \frac{2}{3} \sqrt{\frac{2}{5}} \rho_{3,2} + \frac{1}{5\sqrt{3}} \rho_{4,3} & \frac{2}{3} \rho_{2,2} + \frac{8}{15} \rho_{3,3} + \frac{1}{5} \rho_{4,4} & \cdot \quad \cdot \quad \cdot \\ \cdot & \cdot & \cdot \quad \cdot \quad \cdot \\ \cdot & \cdot & \cdot \quad \cdot \quad \cdot \\ \cdot & \cdot & \cdot \quad \cdot \quad \cdot \end{array} \right)$$

Multilevel Atoms

$$\dot{\rho} = -\frac{i}{\hbar}[H_A + V, \rho] + (\dot{\rho})_{\text{sp}} \equiv Q$$

$$\rho_{ij} = \sigma_{ij} e^{iK_{ij}t}$$

$$i, j = 1, 2, \dots, 12$$

$$\dot{\sigma}_{ij} = e^{-iK_{ij}t} Q_{ij} - iK_{ij} \sigma_{ij}$$

No explicit time dependence

$$K = \begin{array}{c|c} 1, \dots, 7 & 8, \dots, 12 \\ \hline 0 & -\omega \\ \hline \omega & 0 \end{array}$$

Susceptibility

$$P = \frac{N}{V} \langle d \rangle = \frac{N}{V} \text{Tr}(\rho d)$$

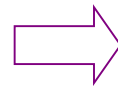
$$= \frac{N}{V} d \sum_{m=-2}^2 C_{F_g=2,m}^{F_e=3,m+1} \sigma_{e_{m+1},g_m} e^{-i\omega t} + c.c$$

$$= 2d \frac{N}{V} \text{Re} \left[\sum_{m=-2}^2 C_{F_g=2,m}^{F_e=3,m+1} \sigma_{e_{m+1},g_m} e^{-i\omega t} \right]$$

$$= \text{Re} \left[\varepsilon_0 \chi E_0 e^{-i\omega t} \right]$$

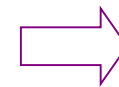
$$\Omega_1 = -\frac{dE_0}{\hbar}$$

$$\Gamma = \frac{1}{3\pi\varepsilon_0} \frac{\omega_0^3}{\hbar c^3} d^2$$



$$\chi = -\frac{N}{V} \frac{3\lambda^3}{4\pi^2} \frac{\Gamma}{\Omega_1} \sum_{m=-2}^2 C_{F_g=2,m}^{F_e=3,m+1} \sigma_{e_{m+1},g_m}$$

Rate equation approximation → Neglect derivative of optical coherence



$$\dot{\sigma}_{ij} = e^{-iC_{ij}t} (\dot{\rho})_{ij} - iC_{ij}\sigma_{ij}$$

$$\dot{\sigma}_{e_i,g_j} \rightarrow 0$$

$$\sigma_{e_{m+1},g_m} = \Omega_1 \frac{C_{g_m}^{e_{m+1}}}{(i\Gamma + 2\delta)} (\sigma_{g_m,g_m} - \sigma_{e_{m+1},e_{m+1}})$$



$$\chi = -\frac{N}{V} \frac{3\lambda^3}{4\pi^2} \sum_{m=-2}^2 (C_{F_g=2,m}^{F_e=3,m+1})^2 \frac{1}{i + (2\delta/\Gamma)} (\sigma_{g_m,g_m} - \sigma_{e_{m+1},e_{m+1}})$$

When the coherences between the magnetic sublevels of the ground or excited states can be neglected

Density matrix equations
→ Rate equations

Averaging over Maxwell-Boltzmann Distribution

1D Maxwell-Boltzmann velocity distribution

$$f_z(v_z)dv_z = \frac{1}{\sqrt{\pi}v_{\text{mp}}} e^{-(v_z/v_{\text{mp}})^2} dv_z$$

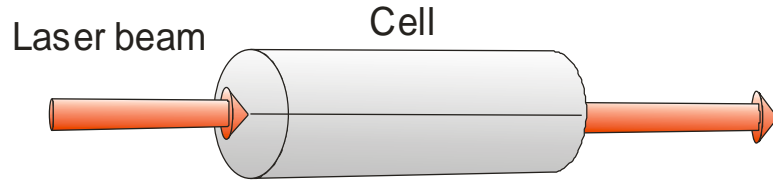
$$v_{\text{mp}} = \sqrt{\frac{2k_B T}{M}} \text{ Most Probable Velocity}$$

Doppler averaging over MB distribution

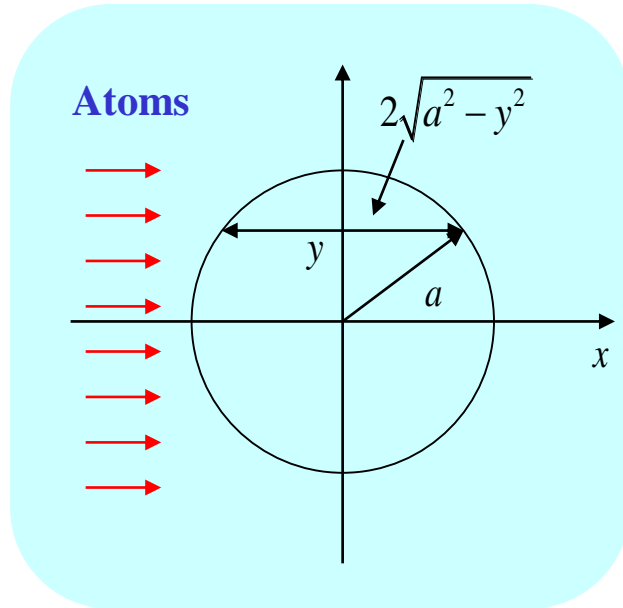
$$\chi(\delta) \rightarrow \int_{-\infty}^{\infty} dv_z \frac{1}{\sqrt{\pi}v_{\text{mp}}} e^{-(v_z/v_{\text{mp}})^2} \chi(\delta_1 \rightarrow \delta - kv_z)$$

Transverse Directions

Doppler Effects for the transverse directions ?



Using average velocity and traveling distance to calculate *average transit time*.



Average distance
$$d_{av} = \frac{1}{a} \int_0^a 2\sqrt{a^2 - y^2} dy = \frac{\pi}{2} a$$

Average velocity in 2D

$$v_{av} = \int_0^\infty v_\perp \frac{1}{\pi v_{mp}^2} 2\pi v_\perp dv_\perp e^{-(v_\perp / v_{mp})^2} = \frac{\sqrt{\pi} v_{mp}}{2}$$

Average transit time

$$t_{av} = \frac{d_{av}}{v_{av}} = \frac{\sqrt{\pi} a}{v_{mp}}$$

Solving time dependent coupled differential equations

$$\bar{\chi} = \frac{1}{t_{av}} \int_0^{t_{av}} dt \int_{-\infty}^{\infty} dv f_D(v) \chi(\delta - kv, t)$$

Transit Relaxation

Steady-state calculation

$$\langle e_m | (\dot{\rho}_{\text{sp}}) | e_{m'} \rangle = -\Gamma \langle e_m | \rho | e_{m'} \rangle - \Gamma_t \langle e_m | \rho | e_{m'} \rangle$$

Not important

$$\langle e_m | (\dot{\rho}_{\text{sp}}) | g_{m'} \rangle = -\frac{\Gamma}{2} \langle e_m | \rho | g_{m'} \rangle$$

$$\langle g_m | (\dot{\rho}_{\text{sp}}) | e_{m'} \rangle = -\frac{\Gamma}{2} \langle g_m | \rho | e_{m'} \rangle$$

$$\langle g_m | (\dot{\rho}_{\text{sp}}) | g_{m'} \rangle = \Gamma \sum_{q=-1}^1 C_{F_g=2,m}^{F_e=3,m+q} C_{F_g=2,m'}^{F_e=3,m'+q} \langle g_m | \rho | g_{m'} \rangle$$

$$- \Gamma_t \left(\langle g_m | \rho | g_{m'} \rangle - \frac{1}{8} \delta_{m,m'} \right)$$

Important

$$\Gamma_t = \frac{1}{t_{\text{av}}} = \frac{v_{\text{mp}}}{\sqrt{\pi} a} \quad \text{Transit decay rate}$$

Steady-state calculation

$$\dot{h}_{ij} = e^{-ic_{ij}t} Q_{ij} - ic_{ij} h_{ij}$$

Averaged Susceptibility

Temporal calculation

$$\bar{\chi} = \frac{1}{t_{av}} \int_0^{t_{av}} dt \int_{-\infty}^{\infty} dv f_D(v) \chi(\delta - kv, t)$$

Steady-state calculation

$$\bar{\chi} = \int_{-\infty}^{\infty} dv f_D(v) \chi(\delta - kv, t)$$

$$\chi = -\frac{N}{V} \frac{3\lambda^3}{4\pi^2} \frac{\Gamma}{\Omega_1} \sum_{m=-2}^2 C_{F_g=2,m}^{F_e=3,m+1} \sigma_{e_{m+1},g_m}$$

Rate Equations

σ^+ polarization

$$P_2^m = \sigma_{g_m, g_m}$$

$$R_{F_g, m_{F_g}}^{F_e, m_{F_e}} = (C_{F_g, m_{F_g}}^{F_e, m_{F_e}})^2$$

$$Q_3^m = \sigma_{e_m, e_m}$$

Rate Equations

$$\dot{P}_2^m = -\frac{\Gamma}{2} R_{2,m}^{3,m+1} s (P_2^m - Q_3^{m+1}) + \sum_{m_e=m-1}^{m+1} \Gamma R_{2,m}^{3,m_e} Q_3^{m_e}$$

$$\dot{Q}_3^m = \frac{\Gamma}{2} R_{2,m-1}^{3,m} s (P_2^{m-1} - Q_3^m) - \sum_{m_g=m-1}^{m+1} \Gamma R_{2,m_g}^{3,m} Q_3^m$$

$$s = \frac{\Omega_1^2/2}{\delta^2 + \Gamma^2/4}$$

$$= \frac{I}{I_s} \frac{1}{1 + 4\delta^2/\Gamma^2}$$

Full Rate Equations

$$\dot{Q}_{F_e}^m = \frac{\Gamma}{2} R_{2,m-1}^{F_e,m} s_{F_e} (P_2^{m-1} - Q_{F_e}^m)$$

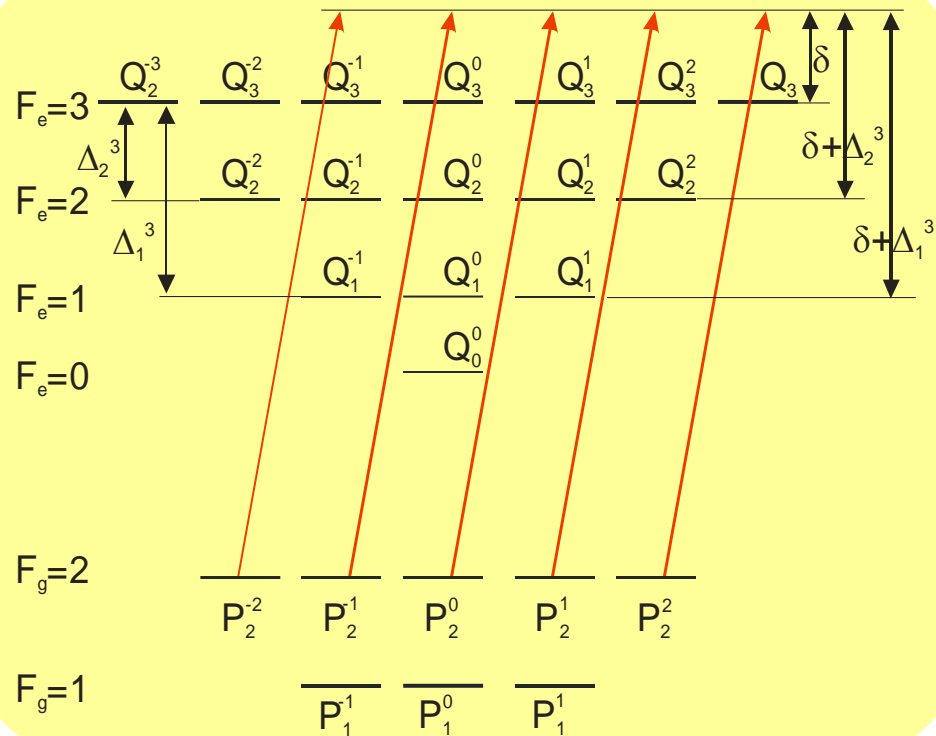
$$- \sum_{F_g=1,2} \sum_{m_e=m-1}^{m+1} \Gamma R_{F_g,m_g}^{F_e,m} Q_{F_e}^m$$

$$F_e = 1, 2, 3$$

$$\dot{P}_2^m = \sum_{F_e=1}^3 \left[-\frac{\Gamma}{2} R_{2,m}^{F_e,m+1} s_{F_e} (P_2^m - Q_{F_e}^{m+1}) \right. \\ \left. + \sum_{m_e=m-1}^{m+1} \Gamma R_{2,m}^{F_e,m_e} Q_{F_e}^{m_e} \right]$$

$$\dot{P}_1^m = \sum_{F_e=0}^2 \sum_{m_e=m-1}^{m+1} \Gamma R_{1,m}^{F_e,m_e} Q_{F_e}^{m_e}$$

$$\sum_{m=-1}^1 P_1^m + \sum_{m=-2}^2 P_2^m + \sum_{m=0}^0 Q_0^m + \sum_{m=-1}^1 Q_1^m + \sum_{m=-2}^2 Q_2^m + \sum_{m=-3}^3 Q_3^m = 1$$



$$s_3 = \frac{I/I_s}{1 + 4\delta^2/\Gamma^2}$$

$$s_2 = \frac{I/I_s}{1 + 4(\delta + \Delta_2^3)^2/\Gamma^2}$$

$$s_1 = \frac{I/I_s}{1 + 4(\delta + \Delta_1^3)^2/\Gamma^2}$$

Contents

- **Two-Level Atoms**

- One field (Optical Bloch equations)
- One field (Linear absorption)
- One field (Analytical solutions to Optical Bloch equations)
- Two fields (Susceptibility)
- Two fields (Polarization spectroscopy for the $F_g=0 \rightarrow F_e=1$ transition)

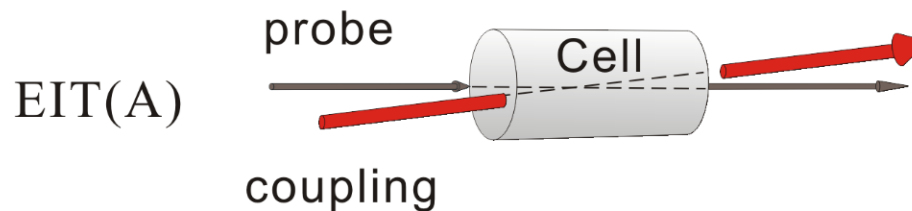
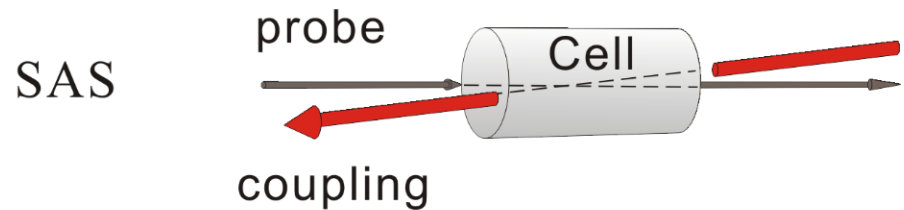
- **Multi-Level Atoms**

- Density matrix equations
- **Electromagnetically induced transparency and absorption**
- Wave mixing and rate equations
- Elliptic polarization and arbitrary quantization axis

EIT and EIA

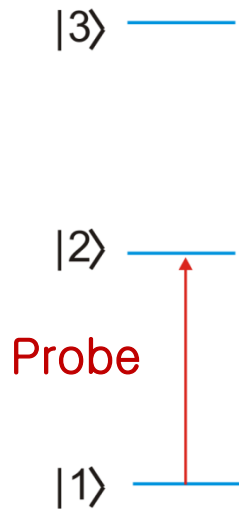
Electromagnetically Induced Transparency and Electromagnetically Induced Absorption

Two Fields

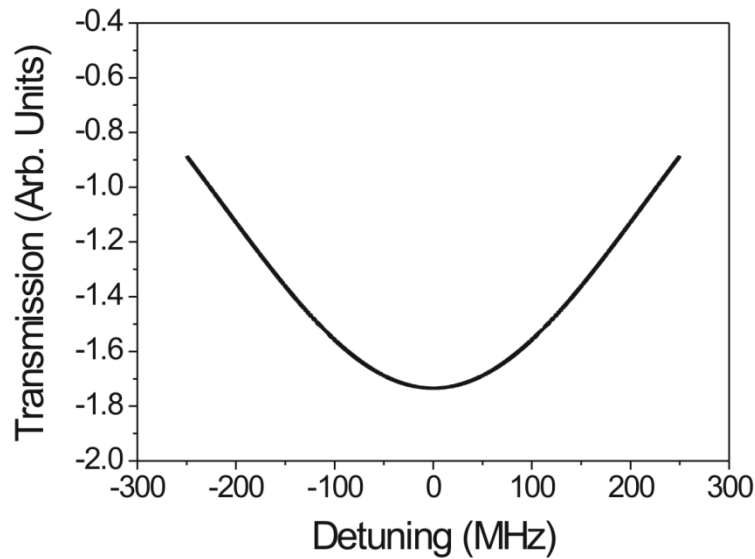


Electromagnetically Induced Transparency

- **Electromagnetically Induced Transparency**



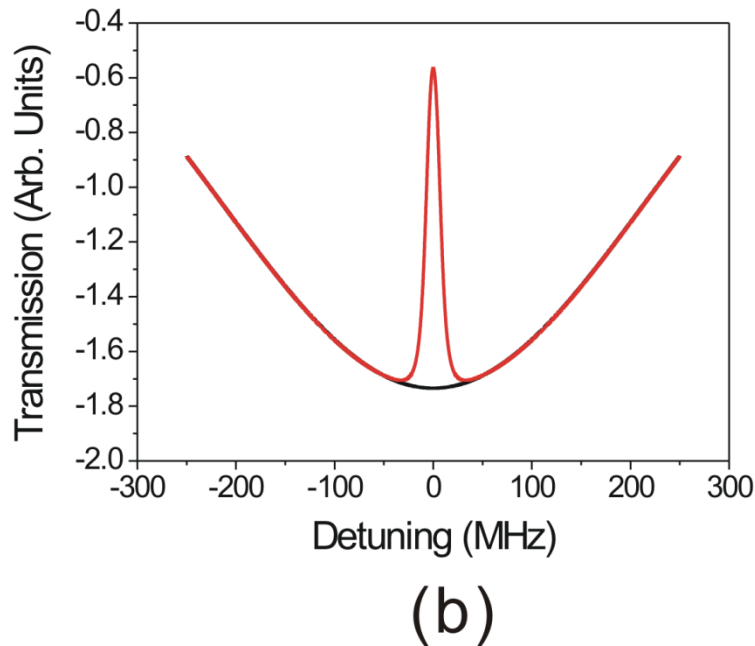
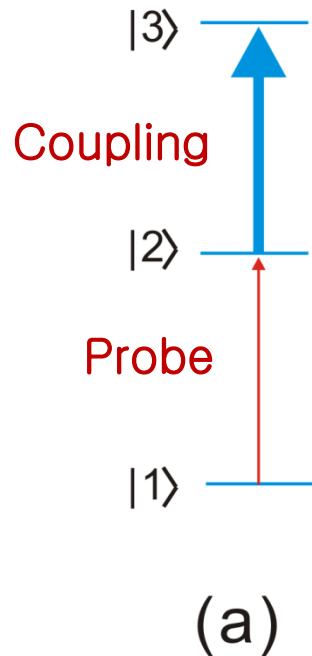
(a)



(b)

Electromagnetically Induced Transparency

- **Electromagnetically Induced Transparency**

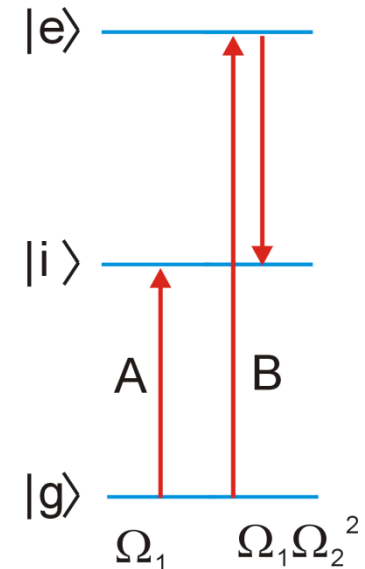


Electromagnetically Induced Transparency

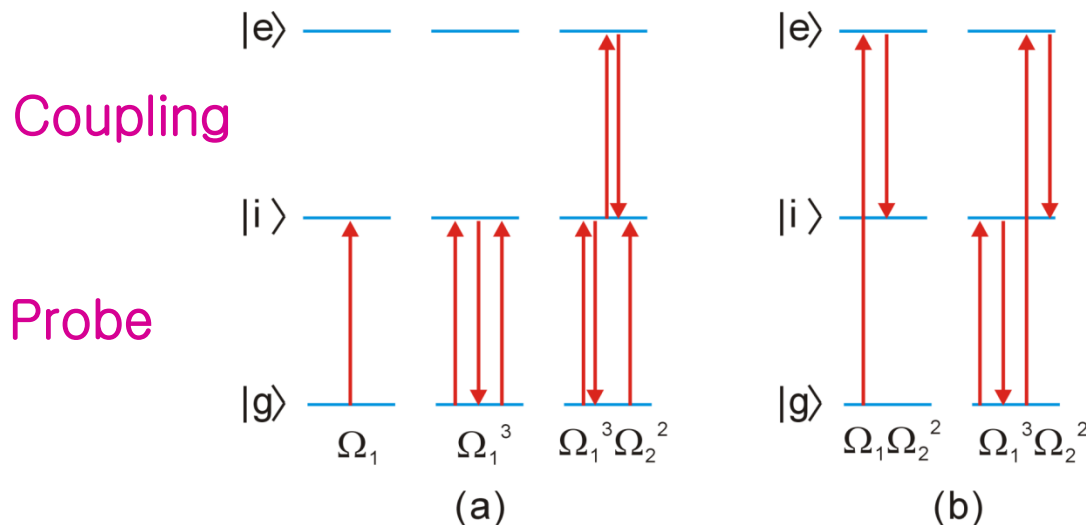
Increase of probe transmission
due to *quantum interference*
between the paths A and B

Electromagnetically Induced Transparency
(EIT)

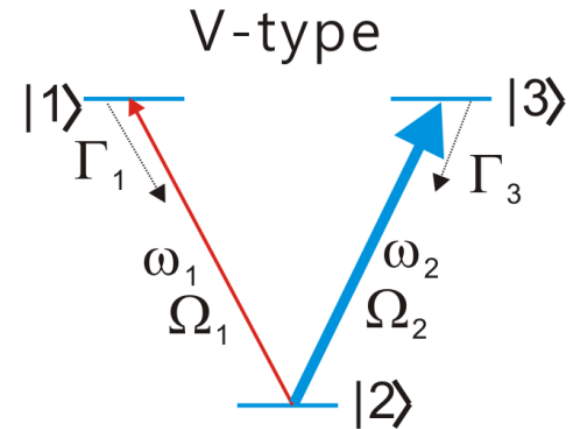
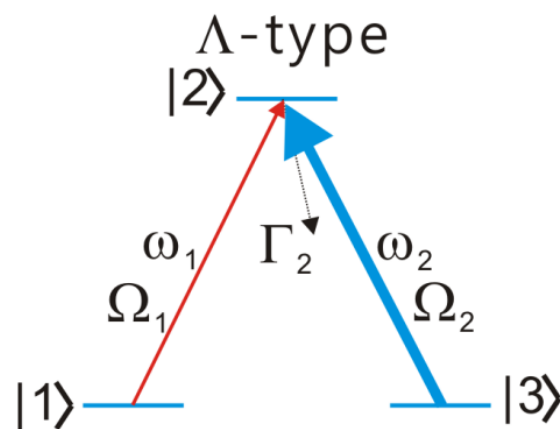
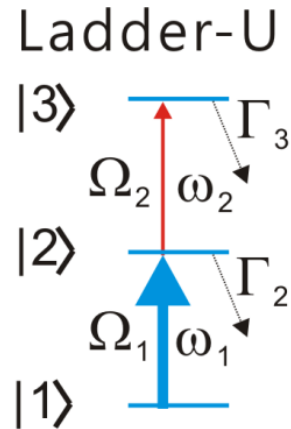
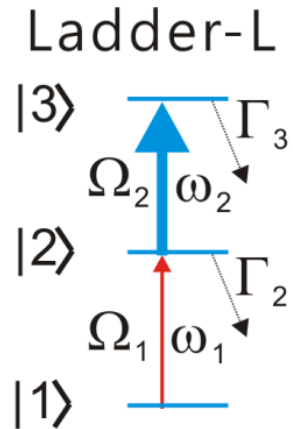
K. J. Boller, et al., Phys. Rev. Lett. **66**, 2593 (1991).



When the intensities are not weak,
there occur *multiphoton* processes



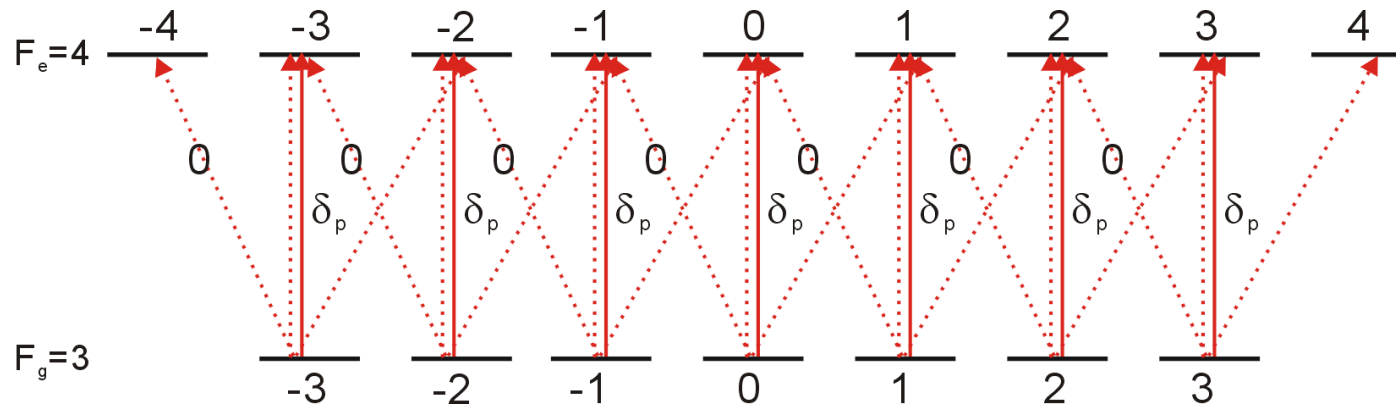
Schemes for EIT



Linewidth for EIT: $\frac{\Gamma_1 + \Gamma_3}{2}$

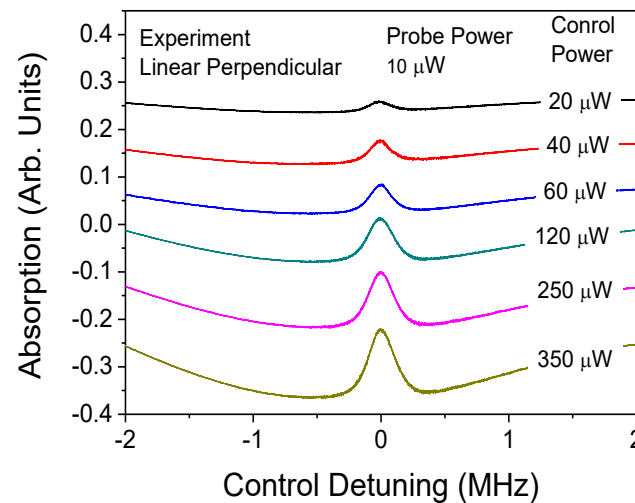
→ Average of the decay rates of the dipole-forbidden transitions

Electromagnetically Induced Absorption (EIA)



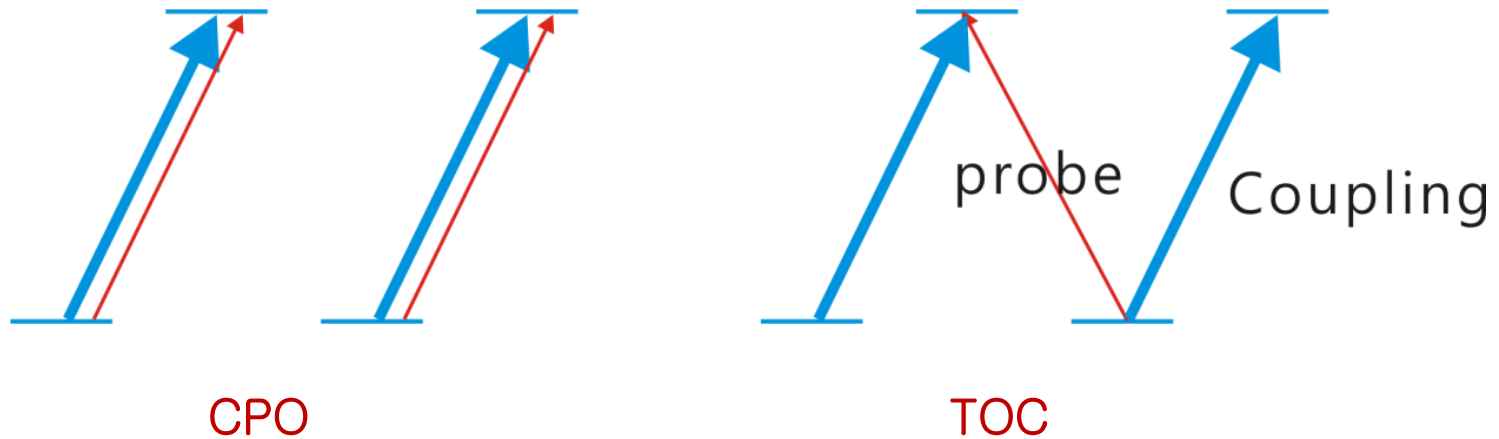
$F_g=3 \rightarrow F_e=4$ transition of ^{85}Rb -D2 line

Polarization configuration (coupling-probe): $\pi \perp \pi$

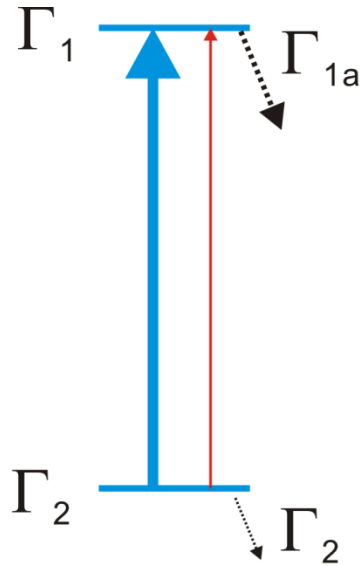


CPO and TOC

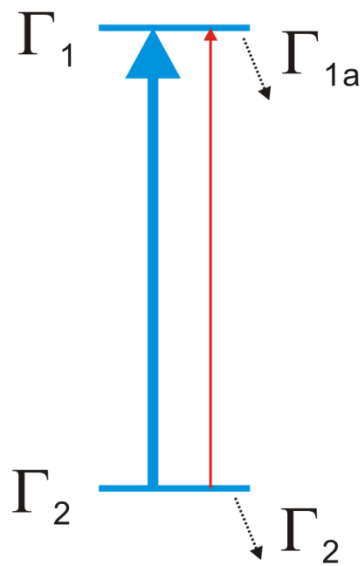
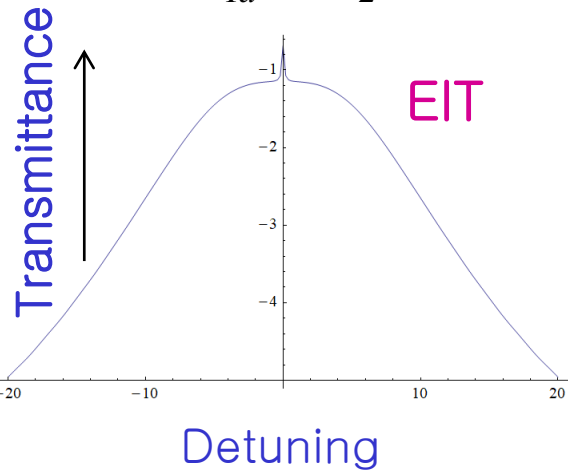
Origin of EIA: Coherent Population Oscillation (CPO)
Transfer Of Coherence (TOC)



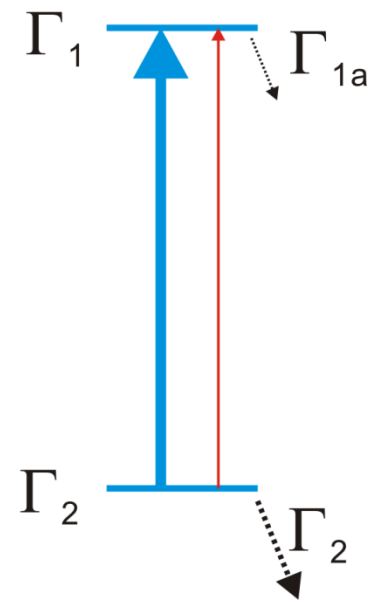
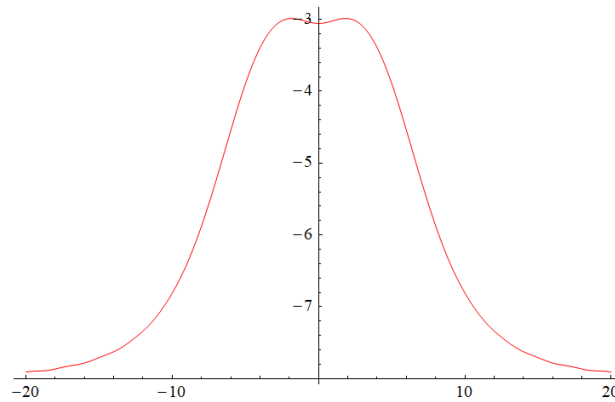
EIT and EIA (CPO)



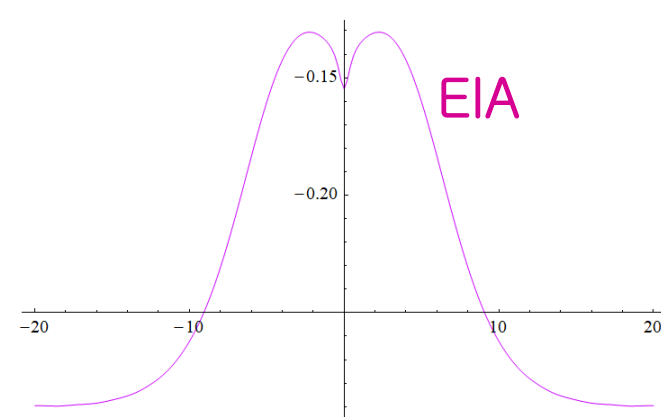
$$\Gamma_{1a} > \Gamma_2$$



$$\Gamma_{1a} = \Gamma_2$$

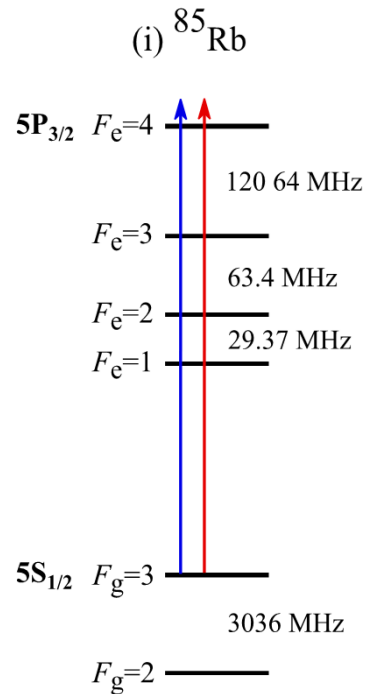
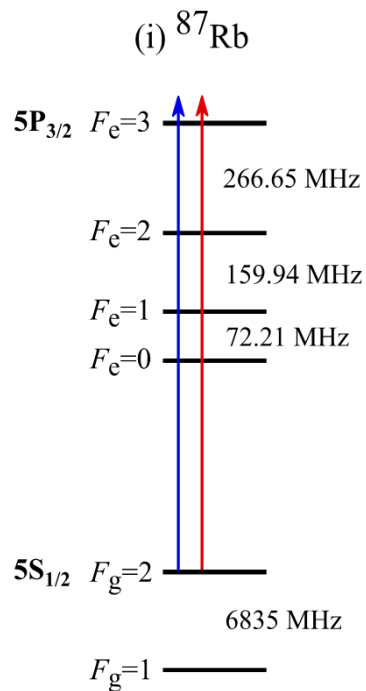


$$\Gamma_{1a} < \Gamma_2$$



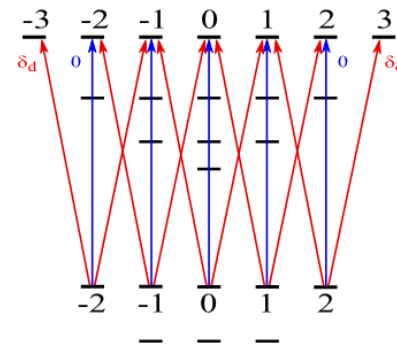
Electromagnetically Induced Absorption

Energy Level Diagram

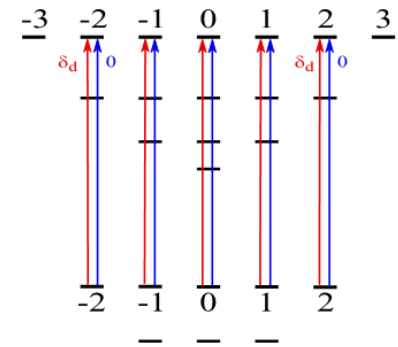


Polarization Schemes

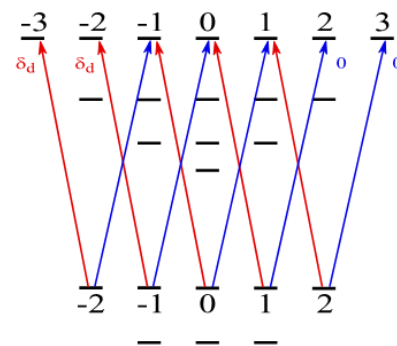
(i) $\pi \perp \pi$



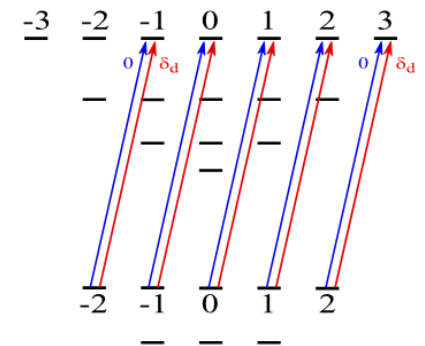
(ii) $\pi \parallel \pi$



(iii) $\sigma \perp \sigma$



(iv) $\sigma \parallel \sigma$



0: coupling beam frequency

δ_d : probe beam frequency

Hamiltonian

^{85}Rb D2 Line $F_g=3 \rightarrow F_e=4$ transition

$$\dot{\rho} = -\frac{i}{\hbar}[H, \rho] + \dot{\rho}_{\text{sp}}$$

$$\begin{aligned} H = & -\sum_{m=-4}^4 \hbar \delta_2 |F_e = 4, m\rangle \langle F_e = 4, m| \\ & + \frac{\hbar}{2} \Omega_1 e^{-i\delta_a t} \sum_{q=-1}^1 a_q \sum_{m=-3}^3 C_{3,m}^{4,m+q} |F_e = 4, m+q\rangle \langle F_g = 3, m| + \text{h.c.} \\ & + \frac{\hbar}{2} \Omega_2 \sum_{q=-1}^1 b_q \sum_{m=-3}^3 C_{3,m}^{4,m+q} |F_e = 4, m+q\rangle \langle F_g = 3, m| + \text{h.c.} \end{aligned}$$

Atomic Hamiltonian

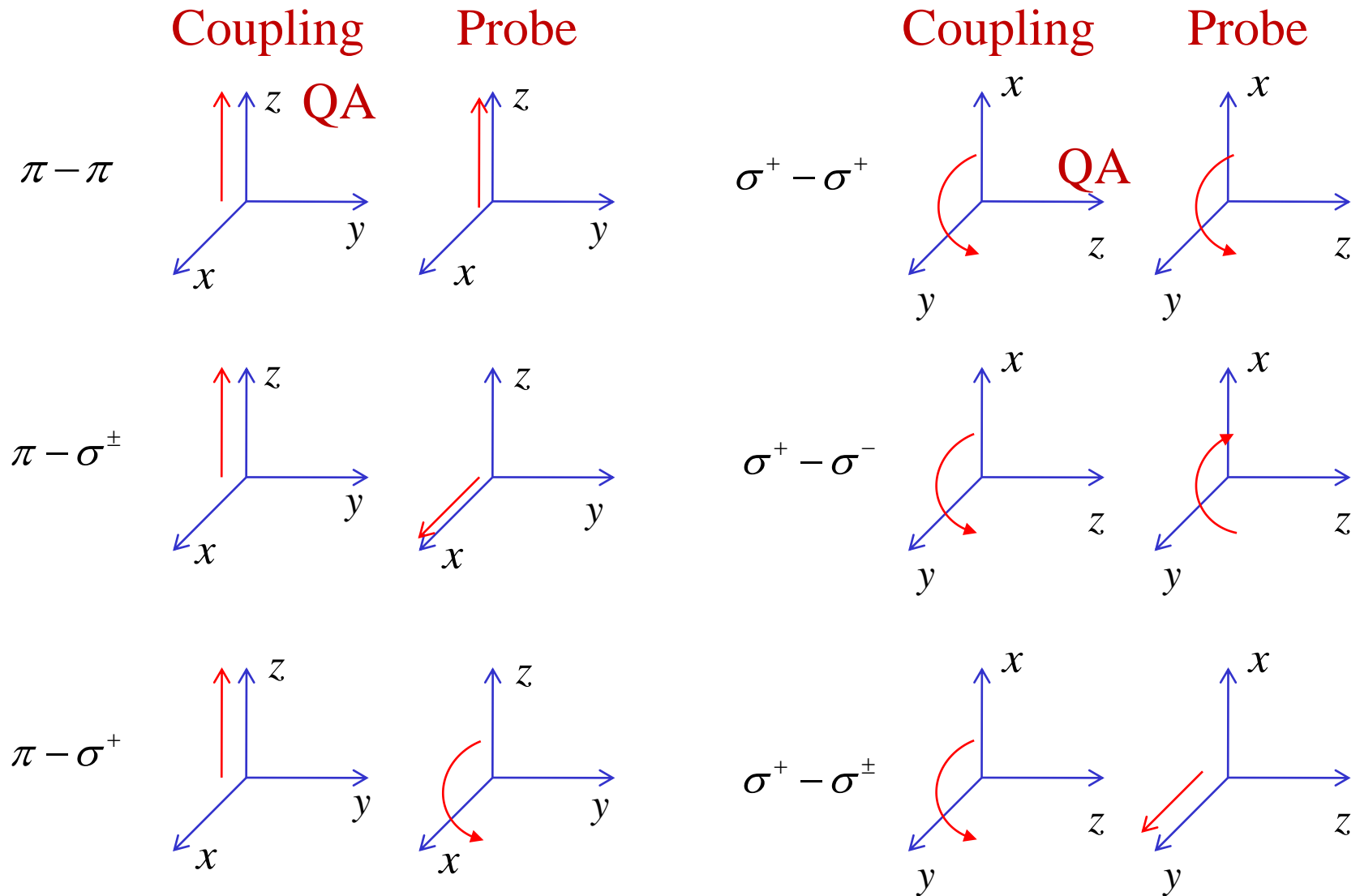
Interaction Hamiltonian
(probe beam)

Interaction Hamiltonian
(coupling beam)

$$\begin{aligned} \langle e_m | \dot{\rho}_{\text{sp}} | e_{m'} \rangle &= -\Gamma \langle e_m | \rho | e_{m'} \rangle \\ \langle e_m | \dot{\rho}_{\text{sp}} | g_{m'} \rangle &= -\frac{\Gamma}{2} \langle e_m | \rho | g_{m'} \rangle \\ \langle g_m | \dot{\rho}_{\text{sp}} | e_{m'} \rangle &= -\frac{\Gamma}{2} \langle g_m | \rho | e_{m'} \rangle \\ \langle g_m | \dot{\rho}_{\text{sp}} | g_{m'} \rangle &= \Gamma \sum_{q=-1}^1 C_m^{m+q} C_{m'}^{m'+q} \langle g_m | \rho | g_{m'} \rangle \\ &\quad - \Gamma_t \left(\langle g_m | \rho | g_{m'} \rangle - \frac{1}{12} \right) \end{aligned}$$

$$\dot{\rho}_{\text{sp}} = \left(\begin{array}{c|c} 1, \dots, 9 & 10, \dots, 16 \\ \hline -\Gamma \rho_{ij} & -\frac{\Gamma}{2} \rho_{ij} \\ \hline -\frac{\Gamma}{2} \rho_{ij} & 7 \times 7 \end{array} \right)$$

Polarization Scheme



Quantization Axis

$$\vec{E} = \hat{\varepsilon} \frac{E}{2} e^{-i\omega_L t} + c.c$$

$$\hat{\varepsilon} = c_+ \hat{\varepsilon}_+ + c_- \hat{\varepsilon}_- + c_0 \hat{\varepsilon}_0$$

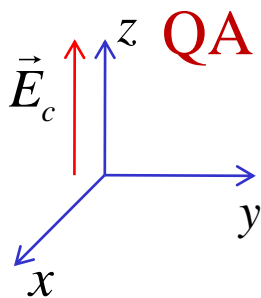
$$\hat{\varepsilon}_- = \frac{1}{\sqrt{2}} (\hat{x} - i\hat{y})$$

$$\hat{\varepsilon}_+ = -\frac{1}{\sqrt{2}} (\hat{x} + i\hat{y})$$

$$\hat{\varepsilon}_0 = \hat{z}$$

$$\longrightarrow \hat{x} = \frac{1}{\sqrt{2}} (\hat{\varepsilon}_- - \hat{\varepsilon}_+)$$

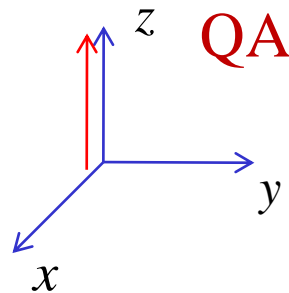
Coupling



$$\hat{\varepsilon}_c = b_{+1} \hat{\varepsilon}_+ + b_{-1} \hat{\varepsilon}_- + b_0 \hat{\varepsilon}_0$$

$$b_{-1} = 0, b_0 = 1, b_{+1} = 0$$

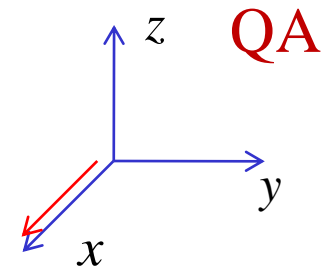
Probe



$$\hat{\varepsilon}_p = a_{+1} \hat{\varepsilon}_+ + a_{-1} \hat{\varepsilon}_- + a_0 \hat{\varepsilon}_0$$

$$a_{-1} = 0, a_0 = 1, a_{+1} = 0$$

Probe



$$\hat{\varepsilon}_p = a_{+1} \hat{\varepsilon}_+ + a_{-1} \hat{\varepsilon}_- + a_0 \hat{\varepsilon}_0$$

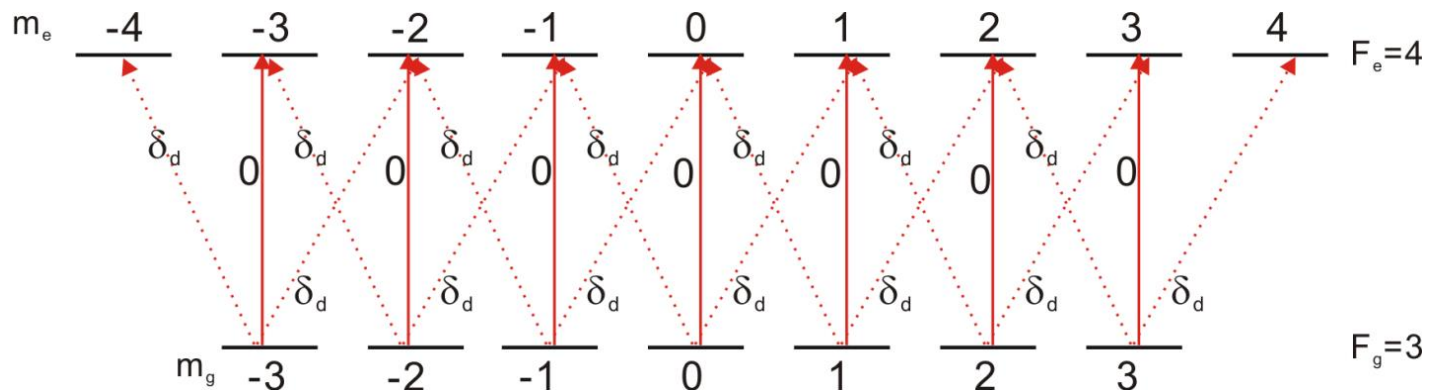
$$a_{-1} = \frac{1}{\sqrt{2}}, a_0 = 0, a_{+1} = -\frac{1}{\sqrt{2}}$$

Oscillation Frequencies

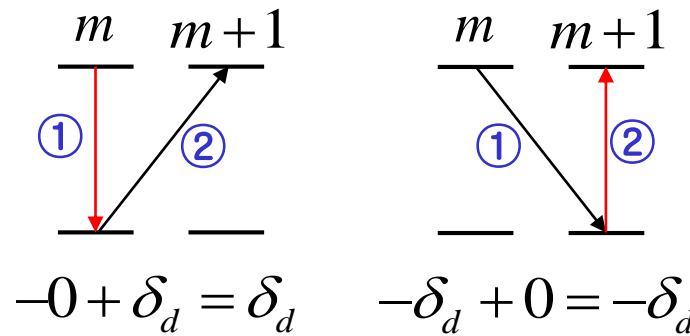
Involving frequencies: ω_1, ω_2 (Probe, coupling)

Involving frequencies relative to ω_2 : $\delta_d, 0$ $\delta_d = \omega_1 - \omega_2 = \delta_1 - \delta_2$

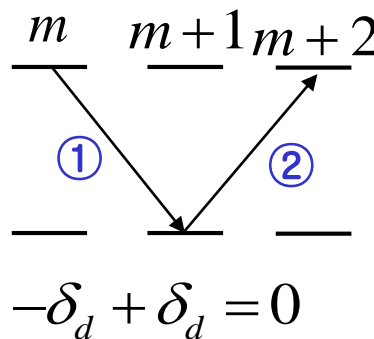
$\pi \perp \pi$ configuration



Oscillation Frequencies (Zeeman Coherence)



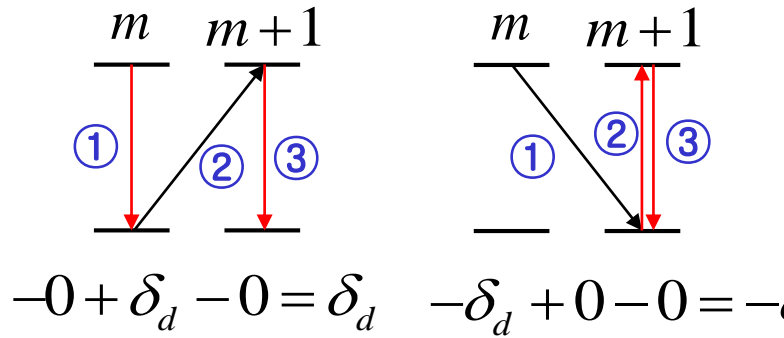
$$\rho_{e_m, e_{m+1}} = \rho_{e_m, e_{m+1}}^{(1)} e^{-i\delta_d t} + \rho_{e_m, e_{m+1}}^{(2)} e^{i\delta_d t}$$



$$\rho_{e_m, e_{m+2}} = \rho_{e_m, e_{m+2}}^{(1)}$$

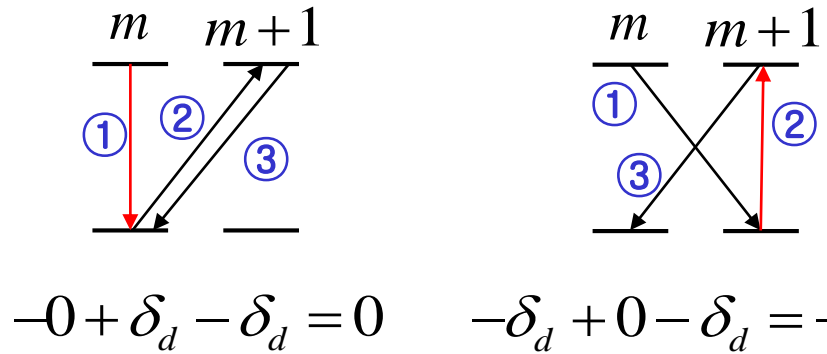
$$\begin{aligned}\rho_{e_m, e_{m+1}} &= \rho_{e_m, e_{m+1}}^{(1)} e^{-i\delta_d t} + \rho_{e_m, e_{m+1}}^{(2)} e^{i\delta_d t}, \\ \rho_{e_m, e_{m+2}} &= \rho_{e_m, e_{m+2}}^{(1)}, \\ \rho_{g_m, g_{m+1}} &= \rho_{g_m, g_{m+1}}^{(1)} e^{-i\delta_d t} + \rho_{g_m, g_{m+1}}^{(2)} e^{i\delta_d t}, \\ \rho_{g_m, g_{m+2}} &= \rho_{g_m, g_{m+2}}^{(1)},\end{aligned}$$

Oscillation Frequencies (Optical Coherence)



$$\rho_{e_m, g_{m+1}} = \rho_{e_m, g_{m+1}}^{(1)} e^{-i\delta_d t} + \rho_{e_m, g_{m+1}}^{(2)} e^{i\delta_d t}$$

$-0 + \delta_d - 0 = \delta_d$ $-\delta_d + 0 - 0 = -\delta_d$

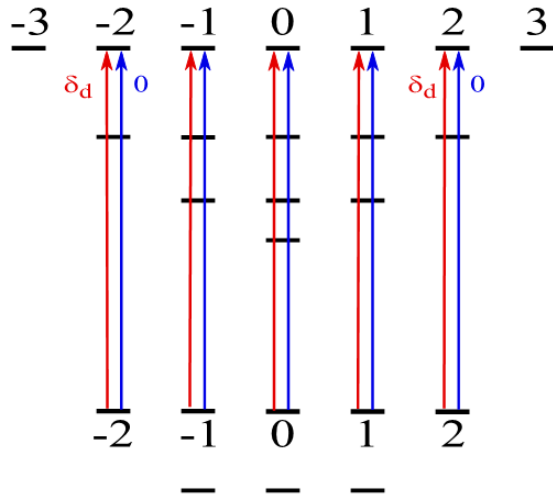


$$\rho_{e_m, g_m} = \rho_{e_m, g_m}^{(1)} + \rho_{e_m, g_m}^{(2)} e^{-2i\delta_d t}$$

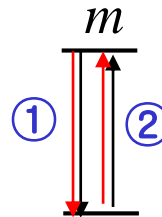
$-0 + \delta_d - \delta_d = 0$ $-\delta_d + 0 - \delta_d = -2\delta_d$

$$\begin{aligned} \rho_{e_m, g_m} &= \rho_{e_m, g_m}^{(1)} + \rho_{e_m, g_m}^{(2)} e^{-2i\delta_d t}, \\ \rho_{e_{m\pm 1}, g_m} &= \rho_{e_{m\pm 1}, g_m}^{(1)} e^{-i\delta_d t} + \rho_{e_{m\pm 1}, g_m}^{(2)} e^{i\delta_d t}, \\ \rho_{e_{m\pm 2}, g_m} &= \rho_{e_{m\pm 2}, g_m}^{(1)} + \rho_{e_{m\pm 2}, g_m}^{(2)} e^{-2i\delta_d t}, \\ \rho_{e_{m\pm 3}, g_m} &= \rho_{e_{m\pm 3}, g_m}^{(1)} e^{-i\delta_d t}, \end{aligned}$$

$\pi \parallel \pi$ Configuration



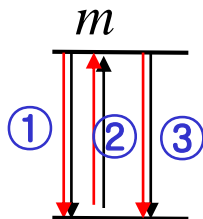
population



$$\{-0, -\delta_d\} \oplus \{0, \delta_d\} = \{0, -\delta_d, \delta_d\}$$

$$\rho_{e_m, e_m} = \rho_{e_m, e_m}^{(0)} + \rho_{e_m, e_m}^{(1)} e^{-i\delta_d t} + \rho_{e_m, e_m}^{(1)*} e^{i\delta_d t}$$

Optical coherence



$$\{-0, -\delta_d\} \oplus \{0, \delta_d\} \oplus \{-0, -\delta_d\}$$

$$= \{0, -\delta_d, \delta_d, -2\delta_d\}$$

$$\rho_{e_m, g_m} = \rho_{e_m, g_m}^{(1)} + \rho_{e_m, g_m}^{(2)} e^{-i\delta_d t} + \rho_{e_m, g_m}^{(3)} e^{i\delta_d t} + \rho_{e_m, g_m}^{(4)} e^{-2i\delta_d t}$$

Calculation

$$\begin{aligned}
 H = & - \sum_{m=-4}^4 \hbar \delta_2 |F_e = 4, m\rangle \langle F_e = 4, m| \\
 & + \frac{\hbar}{2} \Omega_1 e^{-i\delta_d t} \sum_{q=-1}^1 a_q \sum_{m=-3}^3 C_{3,m}^{4,m+q} |F_e = 4, m+q\rangle \langle F_g = 3, m| + \text{h.c.} \\
 & + \frac{\hbar}{2} \Omega_2 \sum_{q=-1}^1 b_q \sum_{m=-3}^3 C_{3,m}^{4,m+q} |F_e = 4, m+q\rangle \langle F_g = 3, m| + \text{h.c.}
 \end{aligned}$$

$$\langle e_m | \dot{\rho}_{\text{sp}} | e_{m'} \rangle = -\Gamma \langle e_m | \rho | e_{m'} \rangle$$

$$\langle e_m | \dot{\rho}_{\text{sp}} | g_{m'} \rangle = -\frac{\Gamma}{2} \langle e_m | \rho | g_{m'} \rangle$$

$$\langle g_m | \dot{\rho}_{\text{sp}} | e_{m'} \rangle = -\frac{\Gamma}{2} \langle g_m | \rho | e_{m'} \rangle$$

$$\langle g_m | \dot{\rho}_{\text{sp}} | g_{m'} \rangle = \Gamma \sum_{q=-1}^1 C_m^{m+q} C_{m'}^{m'+q} \langle g_m | \rho | g_{m'} \rangle - \Gamma_t \left(\langle g_m | \rho | g_{m'} \rangle - \frac{1}{12} \right)$$

$$\dot{\rho} = -\frac{i}{\hbar} [H, \rho] + \dot{\rho}_{\text{sp}}$$

$$\rho_{e_m, e_{m+1}} = \rho_{e_m, e_{m+1}}^{(1)} e^{-i\delta_d t} + \rho_{e_m, e_{m+1}}^{(2)} e^{i\delta_d t},$$

$$\rho_{e_m, e_{m+2}} = \rho_{e_m, e_{m+2}}^{(1)},$$

$$\rho_{g_m, g_{m+1}} = \rho_{g_m, g_{m+1}}^{(1)} e^{-i\delta_d t} + \rho_{g_m, g_{m+1}}^{(2)} e^{i\delta_d t},$$

$$\rho_{g_m, g_{m+2}} = \rho_{g_m, g_{m+2}}^{(1)},$$

$$\rho_{e_m, g_m} = \rho_{e_m, g_m}^{(1)} + \rho_{e_m, g_m}^{(2)} e^{-2i\delta_d t},$$

$$\rho_{e_{m\pm 1}, g_m} = \rho_{e_{m\pm 1}, g_m}^{(1)} e^{-i\delta_d t} + \rho_{e_{m\pm 1}, g_m}^{(2)} e^{i\delta_d t},$$

$$\rho_{e_{m\pm 2}, g_m} = \rho_{e_{m\pm 2}, g_m}^{(1)} + \rho_{e_{m\pm 2}, g_m}^{(2)} e^{-2i\delta_d t},$$

$$\rho_{e_{m\pm 3}, g_m} = \rho_{e_{m\pm 3}, g_m}^{(1)} e^{-i\delta_d t},$$

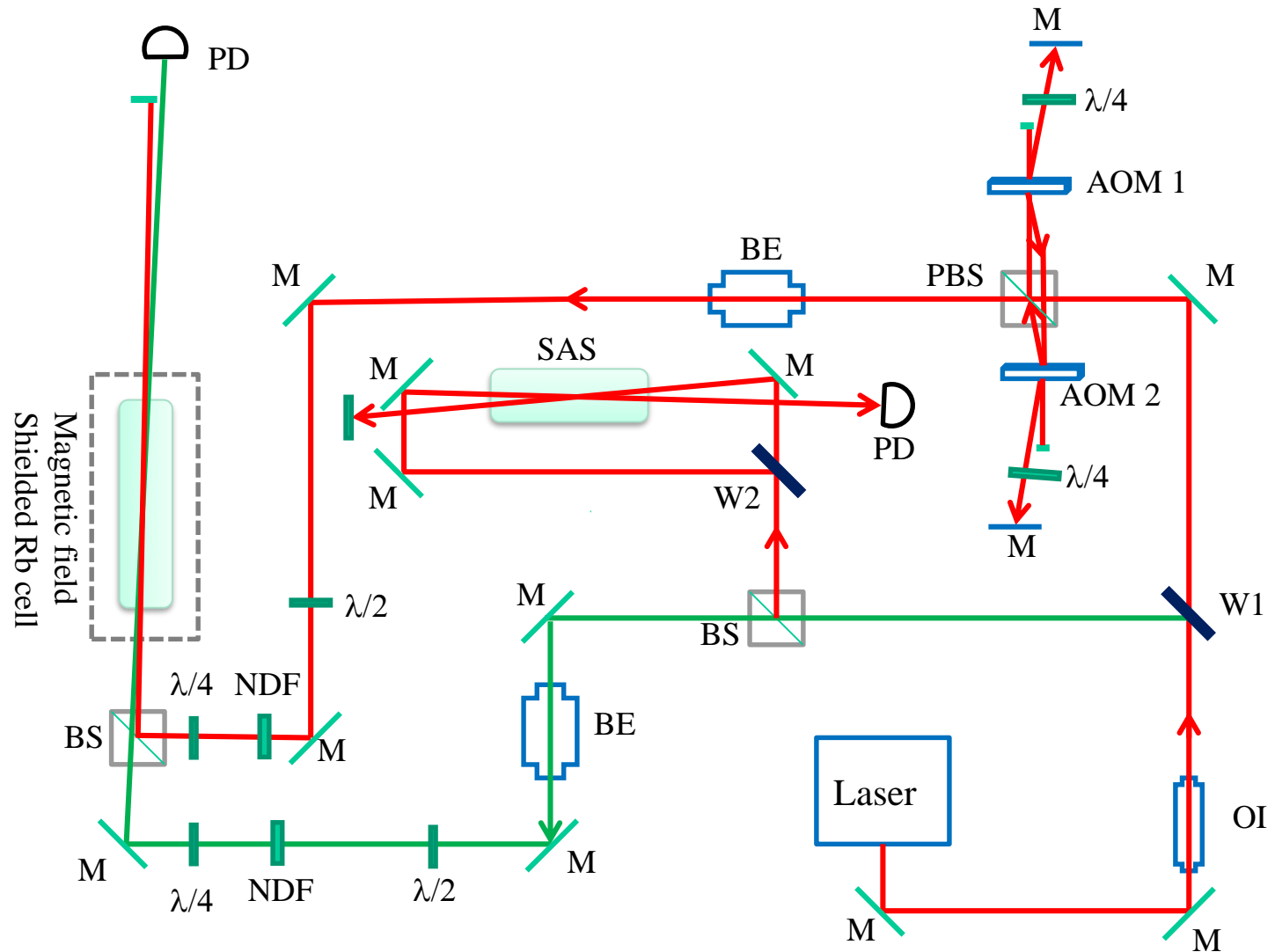
Absorption Coefficient

1. Establish differential equations for each component of the density matrix elements.
2. Calculate $\rho^{(2)}$ as a function of δ , ν in the steady-state regime.

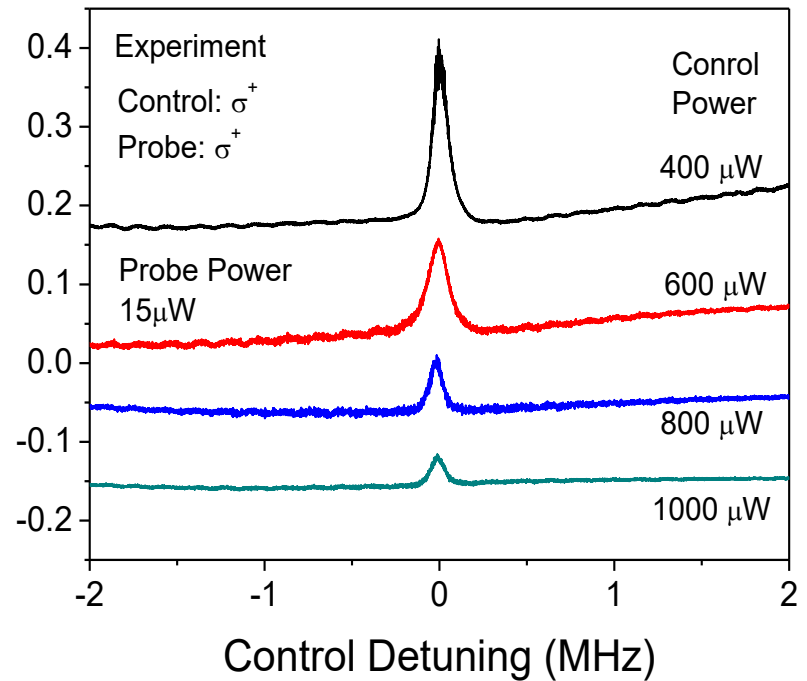
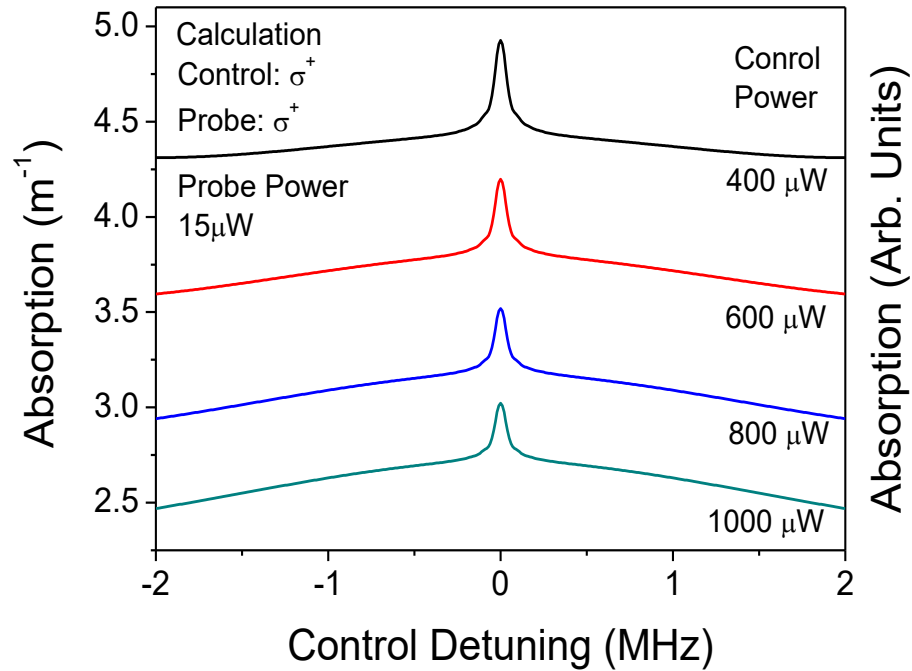
Absorption coefficient for a probe beam

$$\alpha = -\frac{3\lambda^3}{2\pi} \frac{N_{\text{at}}}{\Omega_1} \int_{-\infty}^{\infty} \frac{d\nu}{\sqrt{\pi}\nu_{\text{mp}}} e^{-(\nu/\nu_{\text{mp}})^2} \text{Im} \left[\sum_{q=-1}^1 \sum_{m=-3}^3 a_q^* C_{3,m}^{4,m+q} \rho_{e_{m+q},g_m}^{(2)} \right]$$

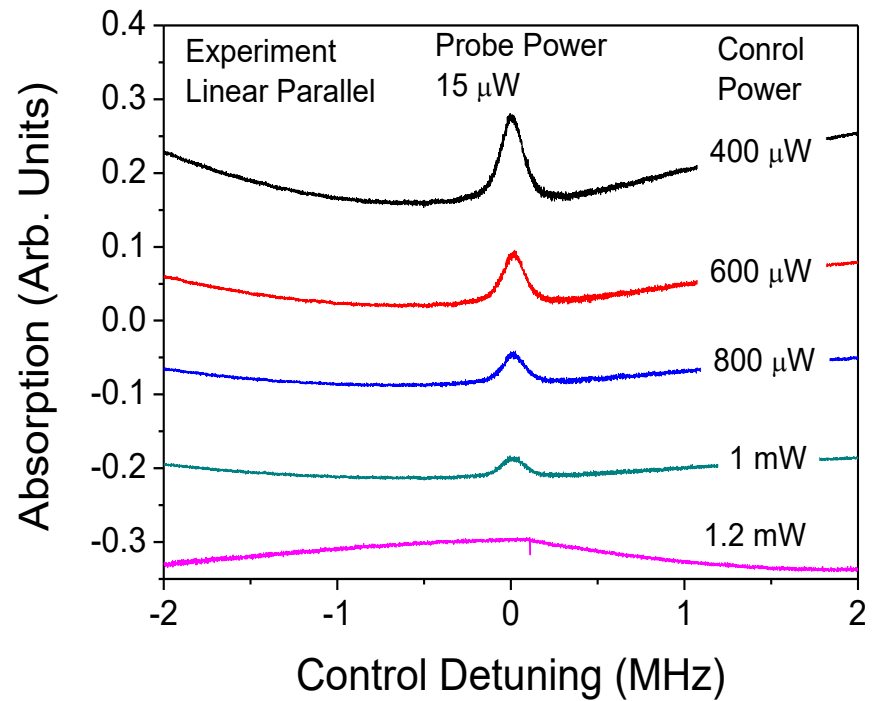
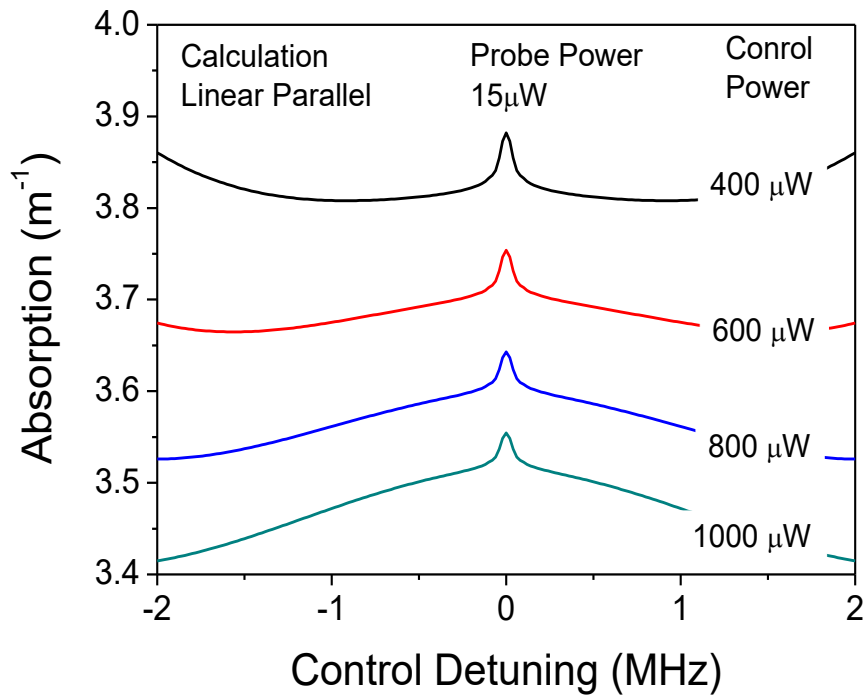
Experimental Setup



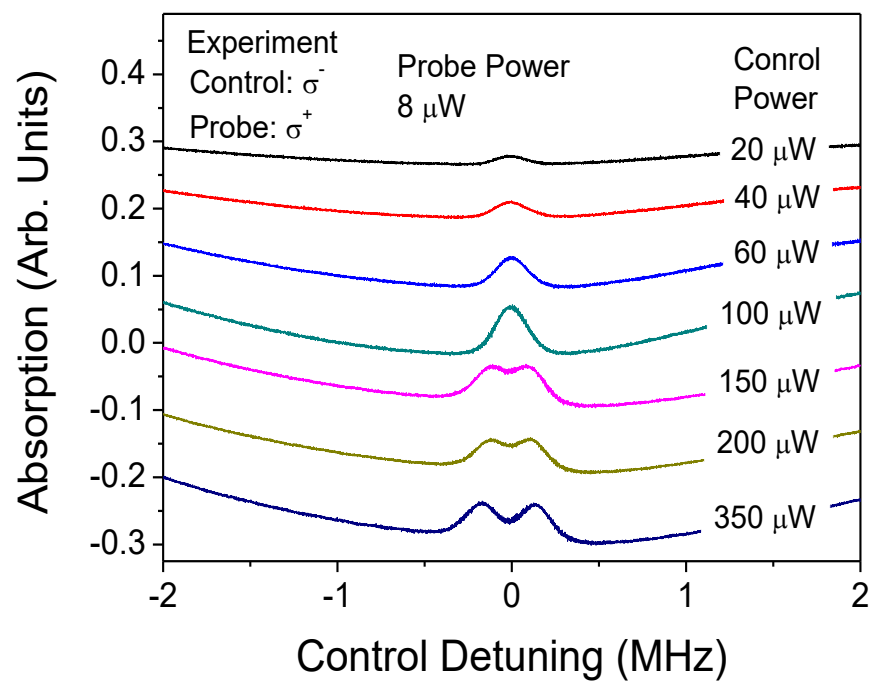
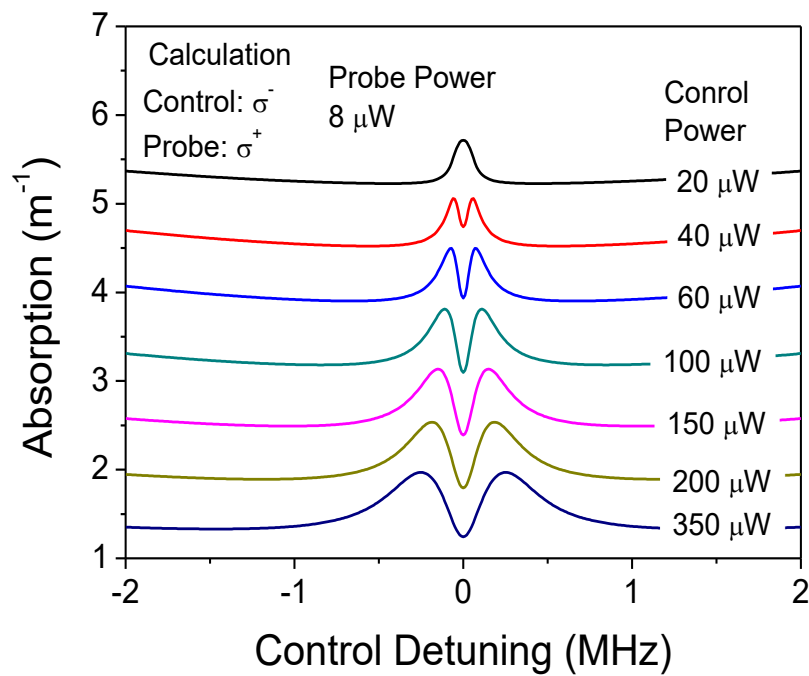
Results ($\sigma \parallel \sigma$)



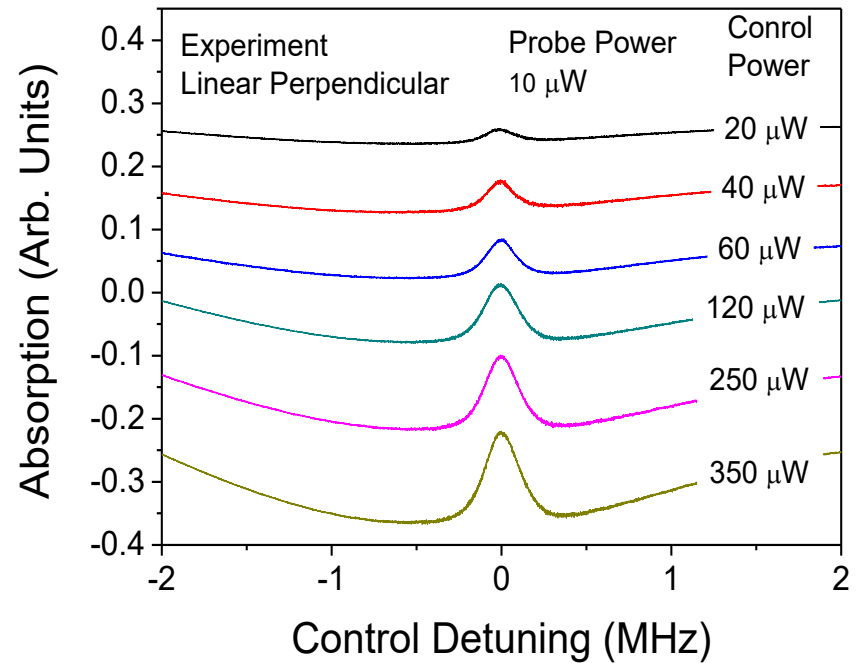
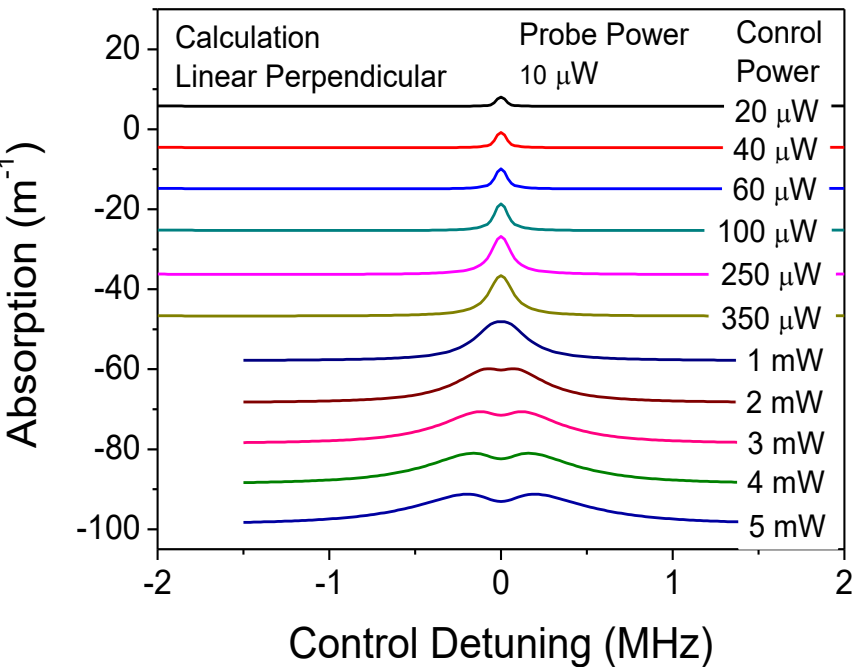
Results ($\pi || \pi$)



Results ($\sigma \perp \sigma$)



Results ($\pi \perp \pi$)



H. Rehman, M. Q. Moshin, H. R. Noh, and J. T. Kim, Opt. Commun. **381**, 127 (2016).

Contents

- **Two-Level Atoms**

- One field (Optical Bloch equations)
- One field (Linear absorption)
- One field (Analytical solutions to Optical Bloch equations)
- Two fields (Susceptibility)
- Two fields (Polarization spectroscopy for the $F_g=0 \rightarrow F_e=1$ transition)

- **Multi-Level Atoms**

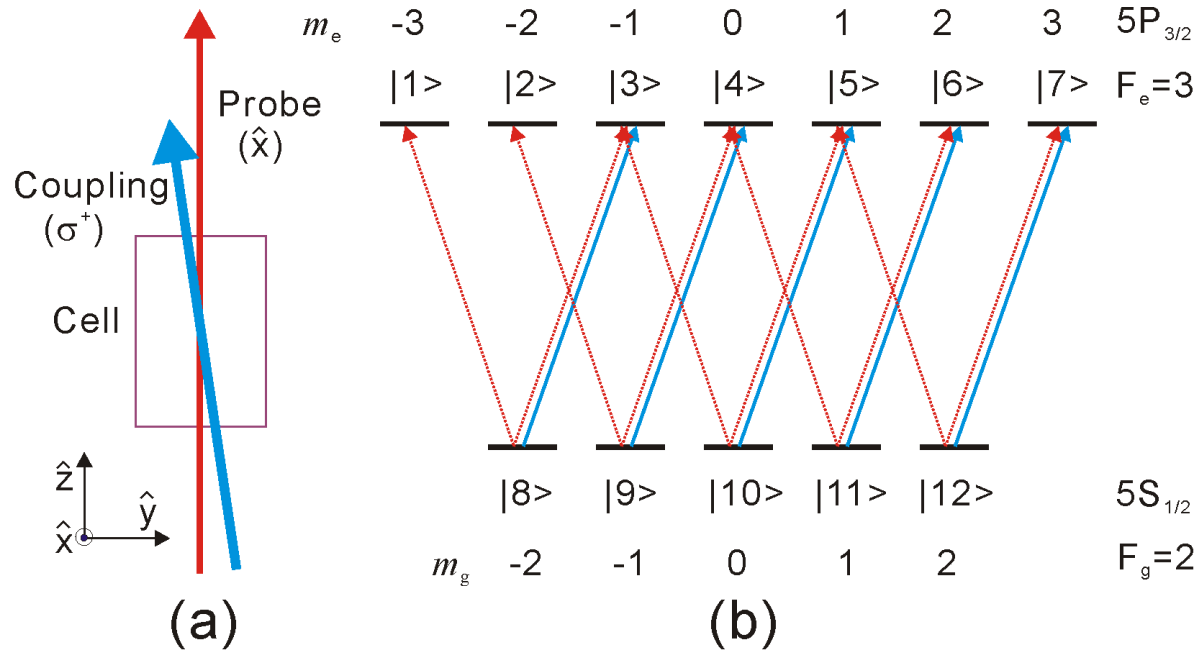
- Density matrix equations
- Electromagnetically induced transparency and absorption
- **Wave mixing and rate equations**
- Elliptic polarization and arbitrary quantization axis

Wave-Mixing and Rate Equations

- Polarization rotation and transmission spectroscopy where the co-propagating probe and coupling beams are linearly and circularly polarized, respectively.
- Calculating the spectra using three methods:
 - (i) calculations using density matrix equations with wave-mixing
 - (ii) calculations using density matrix equations without wave-mixing
 - (iii) calculations using the rate equations.

Theory

^{87}Rb D2 line



Rotation angle

$$\theta = \frac{kL}{4} (\chi_-^r - \chi_+^r),$$

Transmission

$$T = \frac{1}{2} \left(e^{-k\chi_-^i L} + e^{-k\chi_+^i L} \right)$$

Susceptibilities for
 σ^\pm components of
probe

$$\chi_+ = -\frac{3\lambda^3}{4\pi^2} \frac{N_{\text{at}}\Gamma}{c_+\Omega_1} \sum_{j=0}^4 C_{8+j}^{3+j} \int_{-\infty}^{\infty} dv \frac{e^{-v^2/v_{\text{mp}}^2}}{\sqrt{\pi}v_{\text{mp}}} \rho_{3+j,8+j}^{(2)},$$

$$\chi_- = -\frac{3\lambda^3}{4\pi^2} \frac{N_{\text{at}}\Gamma}{c_-\Omega_1} \sum_{j=0}^4 C_{8+j}^{1+j} \int_{-\infty}^{\infty} dv \frac{e^{-v^2/v_{\text{mp}}^2}}{\sqrt{\pi}v_{\text{mp}}} \rho_{1+j,8+j}^{(2)}.$$

Density Matrix Equations

Spontaneous Emission

$$\dot{\rho} = -\frac{i}{\hbar} [H, \rho] + \dot{\rho}_{\text{sp}}$$

Hamiltonian

$$\begin{aligned} H = & -\hbar\delta_2 \sum_{j=1}^7 |j\rangle \langle j| \\ & + \frac{\hbar}{2} (c_+ \Omega_1 e^{-i\delta_d t} + \Omega_2) \sum_{j=0}^4 C_{8+j}^{3+j} |3+j\rangle \langle 8+j| \\ & + \frac{\hbar}{2} c_- \Omega_1 e^{-i\delta_d t} \sum_{j=0}^4 C_{8+j}^{1+j} |1+j\rangle \langle 8+j| + \text{h.c.}, \end{aligned}$$

$$\delta_2 = \delta_c - kv, \quad \delta_d = \delta_p - \delta_c.$$

Probe (Coupling) Rabi

Frequency: Ω_1 (Ω_2)

Normalized transition strength between the ground state $|\mu\rangle$ and excited state $|\nu\rangle$: C_μ^ν

$$\begin{aligned} \langle i | \dot{\rho}_{\text{sp}} | j \rangle &= -[\Gamma + \Gamma_t \delta_{(i,j)}] \rho_{ij}, \quad (i, j) \in (E, E), \\ &= -\frac{\Gamma}{2} \rho_{ij}, \quad (i, j) \in (G, E), \\ &= -\frac{\Gamma}{2} \rho_{ij}, \quad (i, j) \in (E, G), \\ &= \Gamma \sum_{q=-1}^1 C_i^{i-6+q} C_j^{j-6+q} \rho_{i-6+q, j-6+q} \\ &\quad - \Gamma_t \left(\rho_{ii} - \frac{1}{8} \right) \delta_{(i,j)}, \quad (i, j) \in (G, G). \end{aligned}$$

Calculations

(i) Calculations using density matrix equations with wave-mixing

Optical coherences

$$\rho_{\mu,\nu} = \rho_{\mu,\nu}^{(1)} + \rho_{\mu,\nu}^{(2)} e^{-i\delta_a t} + \rho_{\mu,\nu}^{(3)} e^{i\delta_a t} + \rho_{\mu,\nu}^{(4)} e^{-i2\delta_a t},$$

$$(\mu, \nu) = (3, 8), (4, 9), (5, 10), (6, 11), (7, 12),$$

$$(1, 8), (2, 9), (3, 10), (4, 11), (5, 12),$$

$$(5, 8), (6, 9), (7, 10),$$

$$(1, 10), (2, 11), (3, 12).$$

Zeeman coherences

$$\rho_{\mu,\nu} = \rho_{\mu,\nu}^{(1)} + \rho_{\mu,\nu}^{(2)} e^{-i\delta_a t} + \rho_{\mu,\nu}^{(3)} e^{i\delta_a t},$$

$$(\mu, \nu) = (1, 3), (2, 4), (3, 5), (4, 6), (5, 7),$$

$$(8, 10), (9, 11), (10, 12),$$

Populations

$$\rho_{\mu,\mu} = \rho_{\mu,\mu}^{(1)} + \rho_{\mu,\mu}^{(2)} e^{-i\delta_a t} + \rho_{\mu,\mu}^{(2)*} e^{i\delta_a t},$$

$$\mu = 1, 2 \dots, 12.$$

(ii) Calculations using density matrix equations without wave-mixing

Optical coherences

$$\rho_{\mu,\nu} = \rho_{\mu,\nu}^{(1)} + \rho_{\mu,\nu}^{(2)} e^{-i\delta_a t},$$

$$(\mu, \nu) = (3, 8), (4, 9), (5, 10), (6, 11), (7, 12),$$

$$(1, 8), (2, 9), (3, 10), (4, 11), (5, 12),$$

$$(5, 8), (6, 9), (7, 10),$$

$$(1, 10), (2, 11), (3, 12),$$

Zeeman coherences

$$\rho_{\mu,\nu} = \rho_{\mu,\nu}^{(1)},$$

$$(\mu, \nu) = (1, 3), (2, 4), (3, 5), (4, 6), (5, 7),$$

$$(8, 10), (9, 11), (10, 12).$$

Populations

$$\rho_{\mu,\mu} = \rho_{\mu,\mu}^{(1)}, \quad \mu = 1, 2 \dots, 12.$$

(iii) calculations using the rate equations

Optical coherences

$$\rho_{\mu,\nu} = \rho_{\mu,\nu}^{(1)} + \rho_{\mu,\nu}^{(2)} e^{-i\delta_a t},$$

$$(\mu, \nu) = (3, 8), (4, 9), (5, 10), (6, 11), (7, 12),$$

$$(1, 8), (2, 9), (3, 10), (4, 11), (5, 12).$$

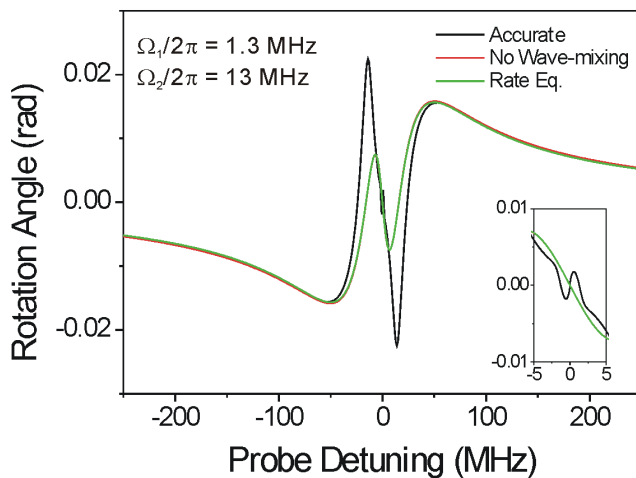
Zeeman coherences \rightarrow Vanish

Populations

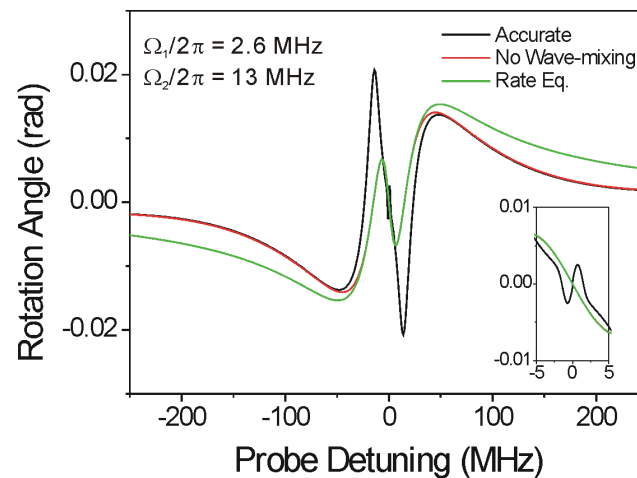
$$\rho_{\mu,\mu} = \rho_{\mu,\mu}^{(1)}, \quad \mu = 1, 2 \dots, 12.$$

Calculated Spectra

Rotation angle

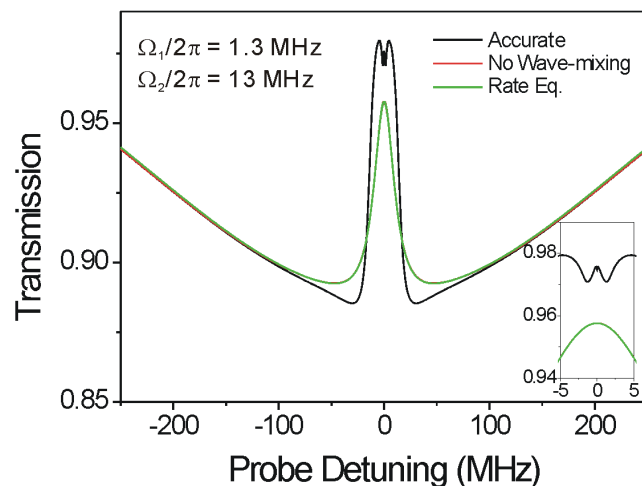


(a)

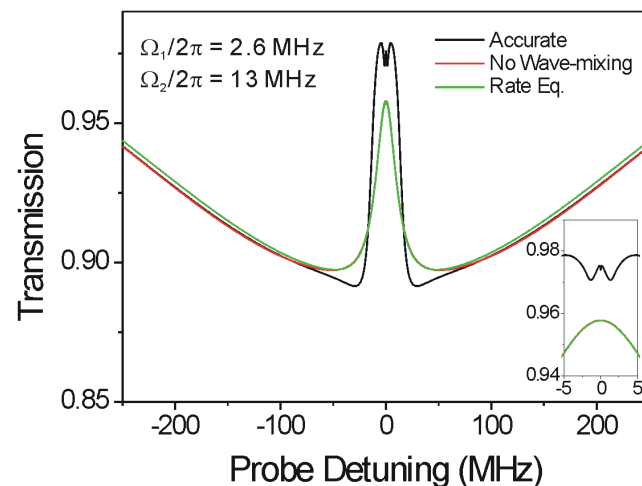


(b)

Transmission



(a)



(b)

Polarization rotation spectra at (a) $\Omega_1/2\pi=1.3$ MHz and (b) $\Omega_1/2\pi=2.6$ MHz. The Rabi frequency of the coupling beam is $\Omega_2/2\pi=13$ MHz.

Contents

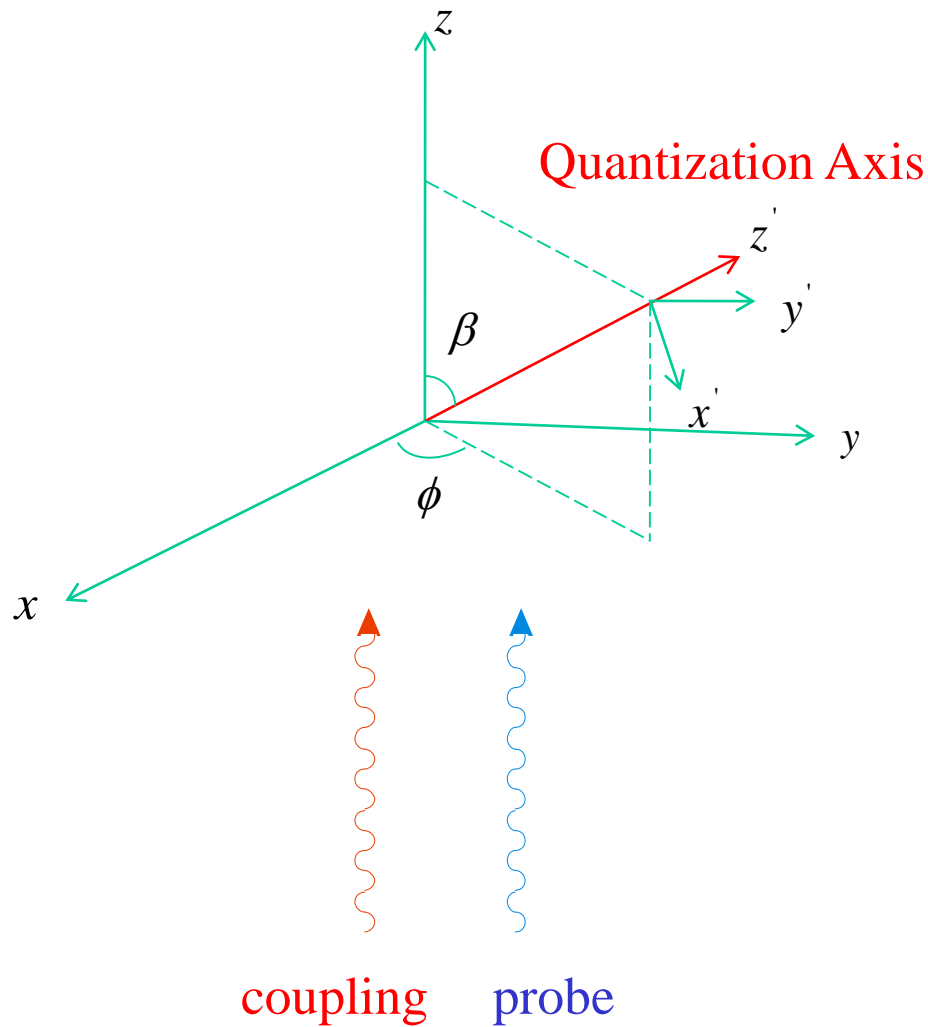
- **Two-Level Atoms**

- One field (Optical Bloch equations)
- One field (Linear absorption)
- One field (Analytical solutions to Optical Bloch equations)
- Two fields (Susceptibility)
- Two fields (Polarization spectroscopy for the $F_g=0 \rightarrow F_e=1$ transition)

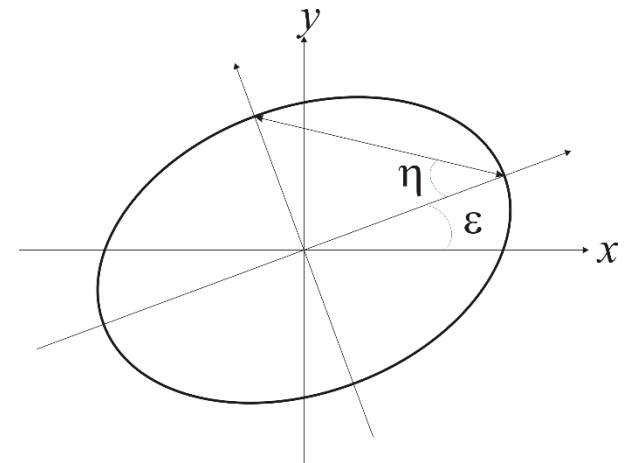
- **Multi-Level Atoms**

- Density matrix equations
- Electromagnetically induced transparency and absorption
- Wave mixing and rate equations
- **Elliptic polarization and arbitrary quantization axis**

Elliptic Polarization and Arbitrary Quantization Axis

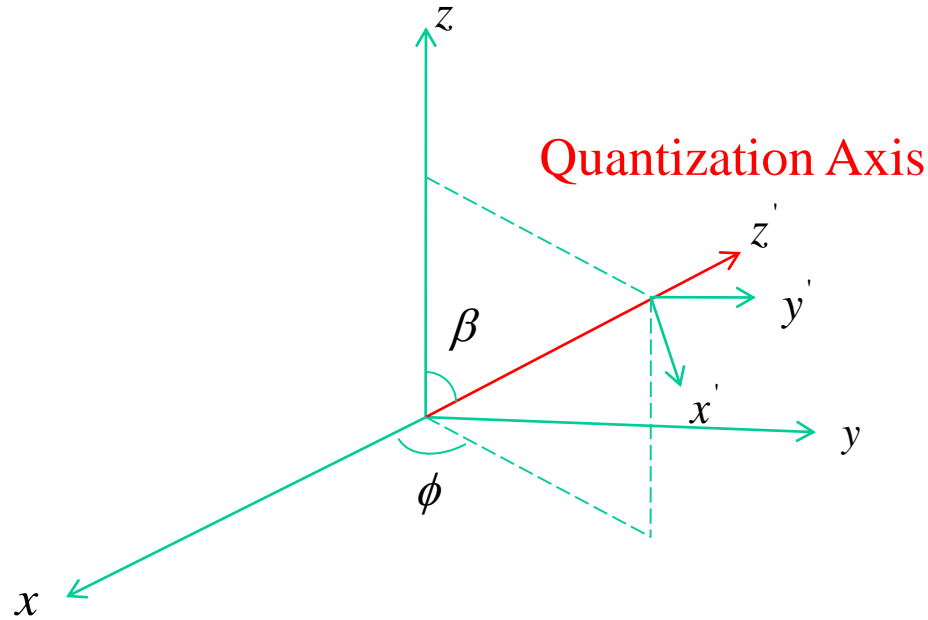


Elliptic polarization



$$E_0 \left\{ -e^{-i\varepsilon} \sin\left(\eta + \frac{\pi}{4}\right) \hat{\varepsilon}_+ - e^{i\varepsilon} \sin\left(\eta - \frac{\pi}{4}\right) \hat{\varepsilon}_- \right\}$$

Coordinate Transformation



$$\hat{z}' = \hat{x} \sin \beta \cos \phi + \hat{y} \sin \beta \sin \phi + \hat{z} \cos \beta$$

$$\hat{x}' = \hat{x} \cos \beta \cos \phi + \hat{y} \cos \beta \sin \phi - \hat{z} \sin \beta$$

$$\hat{y}' = -\hat{x} \sin \phi + \hat{y} \cos \phi$$

$$\hat{\varepsilon}_+ = -\frac{1}{\sqrt{2}}(\hat{x} + i\hat{y})$$

$$\hat{\varepsilon}'_+ = -\frac{1}{\sqrt{2}}(\hat{x}' + i\hat{y}')$$

$$\hat{\varepsilon}_- = \frac{1}{\sqrt{2}}(\hat{x} - i\hat{y})$$

$$\hat{\varepsilon}'_- = \frac{1}{\sqrt{2}}(\hat{x}' - i\hat{y}')$$

$$\hat{\varepsilon}_0 = \hat{z}$$

$$\hat{\varepsilon}'_0 = \hat{z}'$$

Incident electric field

$$E_0 \left\{ -e^{-i\varepsilon} \sin \left(\eta + \frac{\pi}{4} \right) \hat{\varepsilon}_+ - e^{i\varepsilon} \sin \left(\eta - \frac{\pi}{4} \right) \hat{\varepsilon}_- \right\}$$

\Rightarrow

$$\vec{E} = E_0 \{ A_+ \hat{\varepsilon}'_+ + A_0 \hat{\varepsilon}'_0 + A_- \hat{\varepsilon}'_- \}$$

$$A_+ = -\frac{1}{2} e^{i(\phi-\varepsilon)} (1 + \cos \beta) \sin \left(\eta + \frac{\pi}{4} \right) - \frac{1}{2} e^{i(\theta-\varepsilon)} (1 - \cos \beta) \sin \left(\eta - \frac{\pi}{4} \right)$$

$$A_0 = \sin \beta [\cos \eta \cos(\varepsilon - \phi) - i \sin \eta \sin(\varepsilon - \phi)]$$

$$A_- = -\frac{1}{2} e^{i(\phi-\varepsilon)} (1 - \cos \beta) \sin \left(\eta + \frac{\pi}{4} \right) - \frac{1}{2} e^{i(\theta-\varepsilon)} (1 + \cos \beta) \sin \left(\eta - \frac{\pi}{4} \right)$$

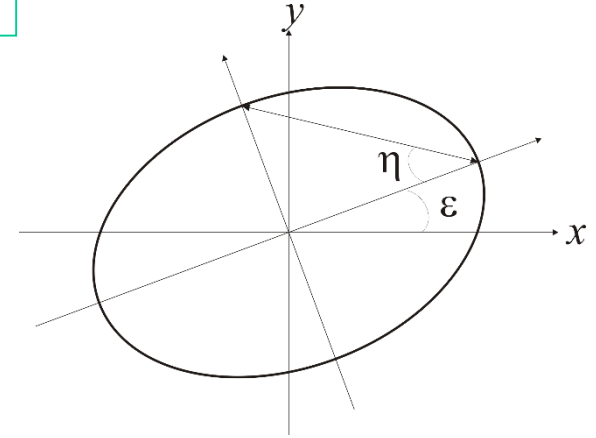
Hamiltonian

^{85}Rb D2 Line $F_g=3 \rightarrow F_e=4$ transition

$$\begin{aligned}
 H = & - \sum_{m=-4}^4 \hbar \delta_2 |F_e = 4, m\rangle \langle F_e = 4, m| \\
 & + \frac{\hbar}{2} \Omega_1 e^{-i\delta_1 t} \sum_{q=-1}^1 a_q \sum_{m=-3}^3 C_{3,m}^{4,m+q} |F_e = 4, m+q\rangle \langle F_g = 3, m| + \text{h.c.} \\
 & + \frac{\hbar}{2} \Omega_2 \sum_{q=-1}^1 b_q \sum_{m=-3}^3 C_{3,m}^{4,m+q} |F_e = 4, m+q\rangle \langle F_g = 3, m| + \text{h.c.}
 \end{aligned}$$

$$\begin{aligned}
 A_+ &= -\frac{1}{2} e^{i(\phi-\varepsilon)} (1 + \cos \beta) \sin\left(\eta + \frac{\pi}{4}\right) - \frac{1}{2} e^{i(\theta-\varepsilon)} (1 - \cos \beta) \sin\left(\eta - \frac{\pi}{4}\right) \\
 A_0 &= \sin \beta [\cos \eta \cos(\varepsilon - \phi) - i \sin \eta \sin(\varepsilon - \phi)] \\
 A_- &= -\frac{1}{2} e^{i(\phi-\varepsilon)} (1 - \cos \beta) \sin\left(\eta + \frac{\pi}{4}\right) - \frac{1}{2} e^{i(\theta-\varepsilon)} (1 + \cos \beta) \sin\left(\eta - \frac{\pi}{4}\right)
 \end{aligned}$$

$$(A_+, A_0, A_-) = \begin{cases} (a_+, a_0, a_-), & \text{probe} \\ (b_+, b_0, b_-), & \text{coupling} \end{cases}$$



Absorption Coefficient

$$\alpha = -\frac{3\lambda^3}{2\pi} \frac{N_{\text{at}}}{\Omega_1} \int_{-\infty}^{\infty} \frac{dv}{\sqrt{\pi} v_{\text{mp}}} e^{-(v/v_{\text{mp}})^2} \text{Im} \left[\sum_{q=-1}^1 \sum_{m=-3}^3 a_q^* C_{3,m}^{4,m+q} \rho_{e_{m+q}, g_m}^{(2)} \right]$$

Conclusions

- **Two-Level Atoms**

- One field (Optical Bloch equations)
- One field (Linear absorption)
- One field (Analytical solutions to Optical Bloch equations)
- **Two fields (Susceptibility)**
- Two fields (Polarization spectroscopy for the $F_g=0 \rightarrow F_e=1$ transition)

- **Multi-Level Atoms**

- Density matrix equations
- Electromagnetically induced transparency and absorption
- Wave mixing and rate equations
- Elliptic polarization and arbitrary quantization axis

THANK YOU FOR YOUR ATTENTION !!