Scattering Amplitudes
Old and New

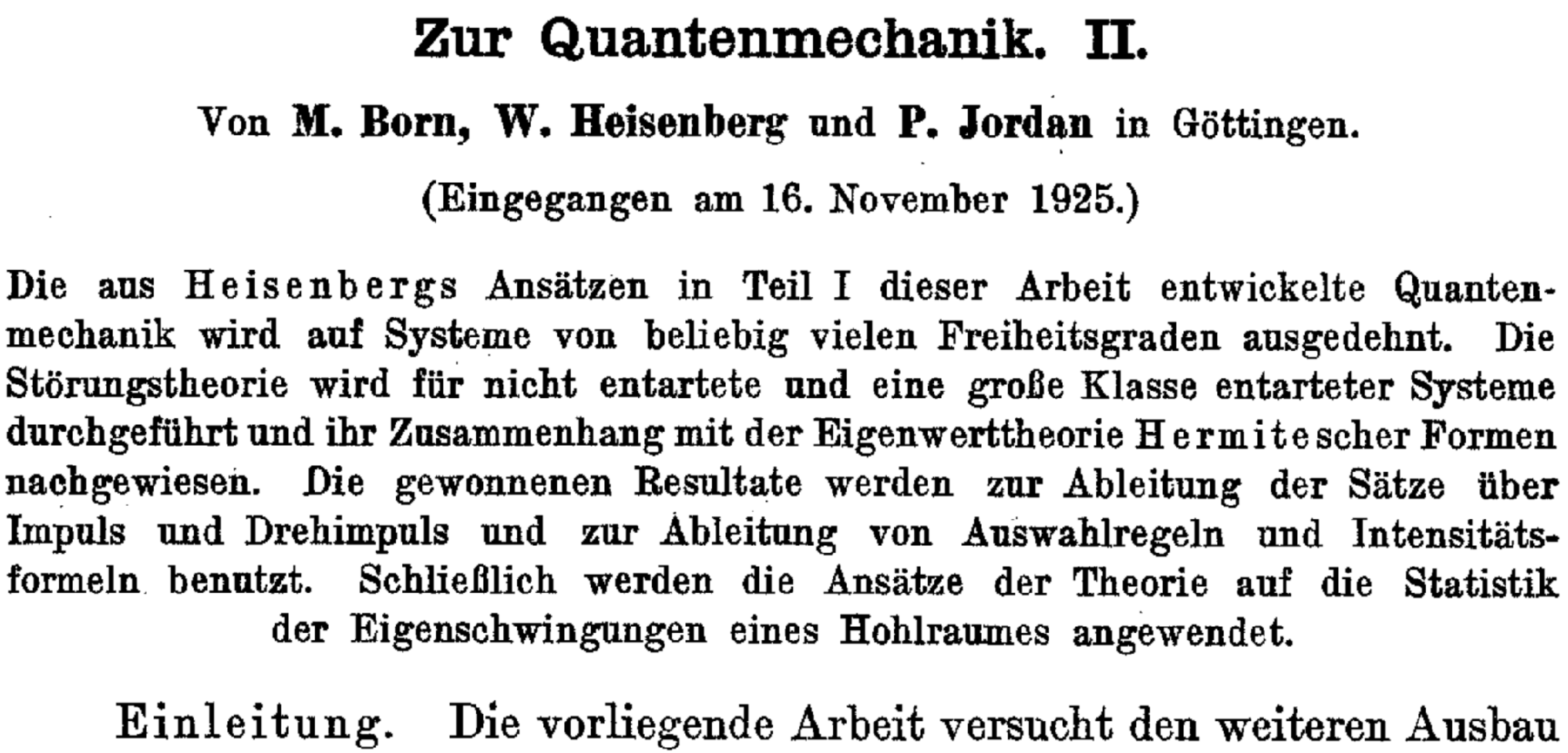
Sangmin Lee

Seoul National University

18 December 2021
KPS-DPF meeting, IBS-CTPU / ZOOM

Introduction

100 years of QFT (almost)

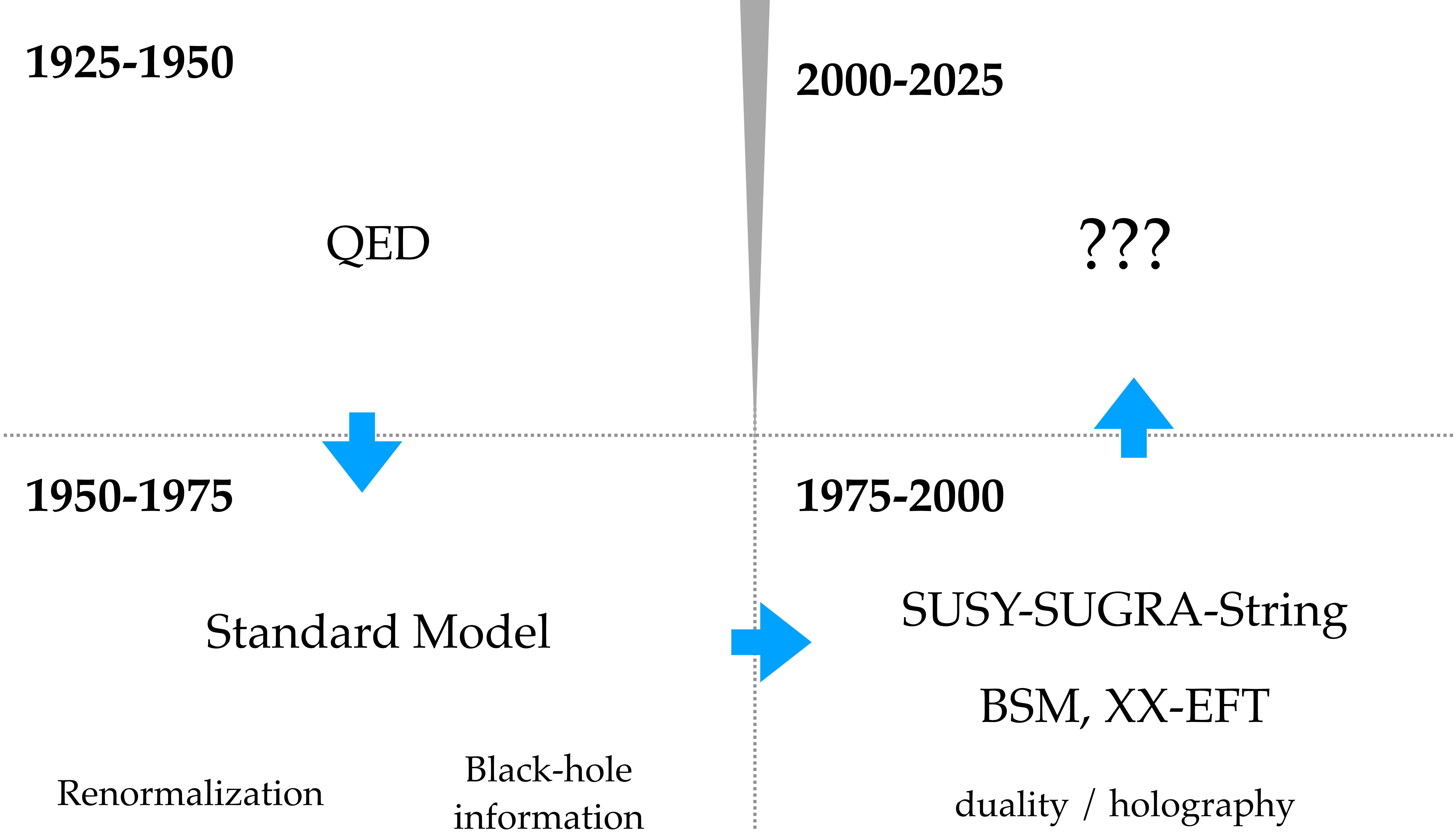

Zur Quantenmechanik. II.

Von M. Born, W. Heisenberg und P. Jordan in Göttingen.

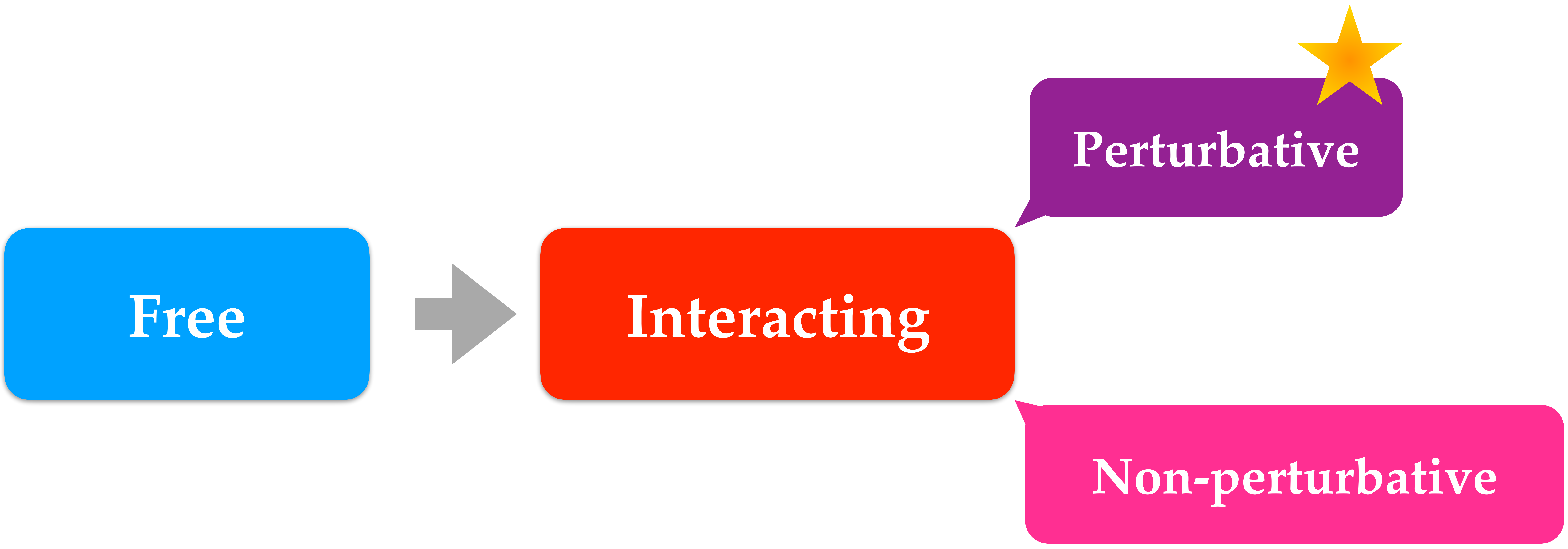
(Eingegangen am 16. November 1925.)

Die aus Heisenbergs Ansätzen in Teil I dieser Arbeit entwickelte Quantenmechanik wird auf Systeme von beliebig vielen Freiheitsgraden ausgedehnt. Die Störungstheorie wird für nicht entartete und eine große Klasse entarteter Systeme durchgeführt und ihr Zusammenhang mit der Eigenwerttheorie Hermitescher Formen nachgewiesen. Die gewonnenen Resultate werden zur Ableitung der Sätze über Impuls und Drehimpuls und zur Ableitung von Auswahlregeln und Intensitätsformeln benutzt. Schließlich werden die Ansätze der Theorie auf die Statistik der Eigenschwingungen eines Hohlraumes angewendet.

Einleitung. Die vorliegende Arbeit versucht den weiteren Ausbau


1925-1950
QED
2000-2025
???
1950-1975
Standard Model
Renormalization
Black-hole information
1975-2000
SUSY-SUGRA-String
BSM, XX-EFT
duality / holography

QFT, oversimplified



Progress in scattering amplitudes of gauge theory and gravity

Simple results

Efficient algorithms

more dramatic for QCD
less so in the electroweak sector

Deeper understanding

Old and new ideas are being unified and recombined !!!

1960's

QFT vs "S-matrix theory"

$$\alpha_f = \sum_i S_{fi}\alpha_i\,, \quad SS^\dagger = 1$$

$$\frac{d\sigma}{d\Omega} \propto |S|^2$$

Unitarity, Locality, Causality

Analyticity

... without QFT ?!

REVIEWS OF MODERN PHYSICS VOLUME 34, NUMBER 3 JULY, 1962

S-Matrix Theory of Strong Interactions without Elementary Particles*†

GEOFFREY F. CHEW
Department of Physics and Lawrence Radiation Laboratory, University of California, Berkeley, California

1. INTRODUCTION

In this paper I present an indecently optimistic view of strong interaction theory. My belief is that a major breakthrough has occurred and that within a relatively short period we are going to achieve a depth of understanding of strong interactions that a few years ago I, at least, did not expect to see within my lifetime. I know that few of you will be convinced by the arguments given here, but I would be masking my feelings if I were to employ a conventionally cautious attitude in this talk. I am bursting with excitement, as are a number of other theorists in this game.

tell me that this is a fetish, that field theory is an equally suitable language, but to me the basic strong-interaction concepts, simple and beautiful in a pure S-matrix approach, are weird, if not impossible, for field theory. It must be said, nevertheless, that my own awareness of these concepts was largely achieved through close collaboration with three great experts in field theory, M. L. Goldberger, Francis Low, and Stanley Mandelstam. Each of them has played a major role in the development of the strong interaction theory that I describe,[1] even though the language of my description may be repugnant to them. Murray Gell-Mann, also, although he has not actumerrymaking lies before us. All the physicists who never learned field theory can get in the game, and

Unitarity, Locality, Causality

Analyticity

... with QFT (and string) !!!

1980's

Perturbative QFT

beyond Feynman diagrams

No more
"shut up and calculate"

Supercollider physics

Reviews of Modern Physics, Vol. 56, No. 4, October 1984

E. Eichten

Fermi National Accelerator Laboratory, P.O. Box 500, Batavia, Illinois 60510

I. Hinchliffe

Lawrence Berkeley Laboratory, Berkeley, California 94720

K. Lane

The Ohio State University, Columbus, Ohio 43210

C. Quigg

Fermi National Accelerator Laboratory, P.O. Box 500, Batavia, Illinois 60510

Eichten *et al.* summarize the motivation for exploring the 1-TeV ($=10^{12}$ eV) energy scale in elementary particle interactions and explore the capabilities of proton-(anti)proton colliders with beam energies between 1 and 50 TeV. The authors calculate the production rates and characteristics for a number of conventional processes, and discuss their intrinsic physics interest as well as their role as backgrounds to more exotic phenomena. The authors review the theoretical motivation and expected signatures for several new phenomena which may occur on the 1-TeV scale. Their results provide a reference point for the choice of machine parameters and for experiment design.

CONTENTS

TeV. From Fig. 78 we find the corresponding two-jet cross section (at $p_\perp = 0.5$ TeV/c) to be about 7×10^{-2} nb/GeV, which is larger by an order of magnitude. Let us next consider the cross section in the neighborhood of the peak in Fig. 102. The integrated cross section in the bin $0.3 \le \cos\theta \le 0.4$ is approximately 0.1 nb/GeV, with transverse energy given roughly by $\langle E_T\rangle \approx (1 \text{ TeV}) \times \langle\cos\theta\rangle = 350$ GeV. The corresponding two-jet cross section, again from Fig. 78, is approximately 10 nb/GeV, which is larger by 2 orders of magnitude. In fact, we have certainly underestimated $\langle E_T\rangle$ and thus somewhat overestimated the two-jet/three-jet ratio in this second case.

We draw two conclusions from this very casual analysis:

At least at small-to-moderate values of E_T, two-jet events should account for most of the cross section.

The three-jet cross section is large enough that a detailed study of this topology should be possible.

It is apparent that these questions are amenable to detailed investigation with the aid of realistic Monte Carlo simulations. Given the elementary two→three cross sections and reasonable parametrizations of the fragmentation functions, this exercise can be carried out with some degree of confidence.

For multijet events containing more than three jets, the theoretical situation is considerably more primitive. A specific question of interest concerns the QCD four-jet background to the detection of W^+W^- pairs in their nonleptonic decays. The cross sections for the elementary two→four processes have not been calculated, and their complexity is such that they may not be evaluated in the foreseeable future. It is worthwhile to seek estimates of the four-jet cross sections, even if these are only reliable in restricted regions of phase space.

Another background source of four-jet events is double parton scattering, as shown in Fig. 103. If all the parton momentum fractions are small, the two interactions may be treated as uncorrelated. The resulting four-jet cross section with transverse energy E_T may then be approximated by

$$\ldots \delta(E_{T1}+E_{T2}-E_T) \,/\, \sigma_{\text{total}} , \quad (3.47)$$

...tes ...nt. ...$\bar{p}pS$ ...ds, ...is

We conclude this section with a brief summary of the ranges of jet energy which are accessible for various beam energies and luminosities. We find essentially no differences between pp and $\bar{p}p$ collisions, so only pp results will be given except at $\sqrt{s}=2$ TeV where $\bar{p}p$ rates are quoted. Figure 104 shows the E_T range which can be explored at the level of at least one event per GeV of E_T per unit rapidity at 90° in the c.m. (compare Figs. 77–79 and 83). The results are presented in terms of the transverse energy per event E_T, which corresponds to twice the transverse momentum $p_\perp$ of a jet. In Fig. 105 we plot the values of E_T that distinguish the regimes in which the two-gluon, quark-gluon, and quark-quark final states are dominant. Comparing with Fig. 104, we find that while the accessible ranges of E_T are impressive, it seems extremely difficult to obtain a clean sample of quark jets. Useful for estimating trigger rates is the total cross section for two jets integrated over $E_T(=2p_\perp) > E_{T_0}$ for both jets in a rapidity interval of -2.5 to $+2.5$. This is shown for pp collisions in Fig. 106.

IV. ELECTROWEAK PHENOMENA

In this section we discuss the supercollider processes associated with the standard model of the weak and electromagnetic interactions (Glashow, 1961; Weinberg, 1967; Salam, 1968). By "standard model" we understand the $SU(2)_L \otimes U(1)_Y$ theory applied to three quark and lepton doublets, and with the gauge symmetry broken by a single complex Higgs doublet. The particles associated with the electroweak interactions are therefore the (left-handed) charged intermediate bosons $W^\pm$, the neutral intermedi-

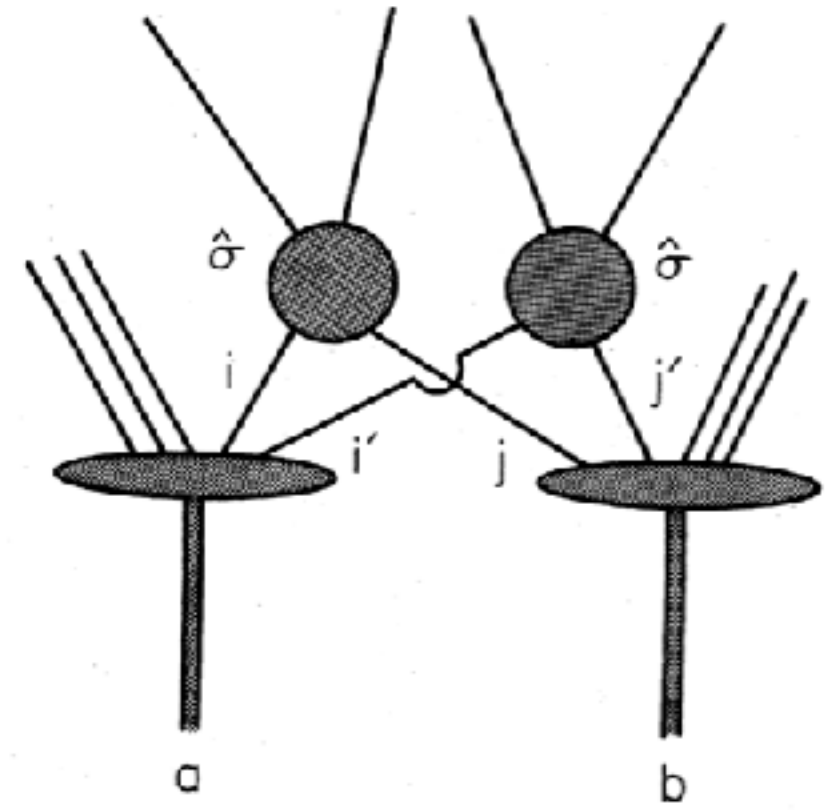


FIG. 103. Four-jet topology arising from two independent parton interactions.

The cross sections for the elementary two→four processes have not been calculated, and their complexity is such that they may not be evaluated in the foreseeable future.

[Jacob Bourjaily, talk 2014]

Nuclear Physics B269 (1986) 410–420

220 Feynman diagrams, thousands of terms.

The final result was summarized in 8 pages.

THE CROSS SECTION FOR FOUR-GLUON PRODUCTION BY GLUON–GLUON FUSION

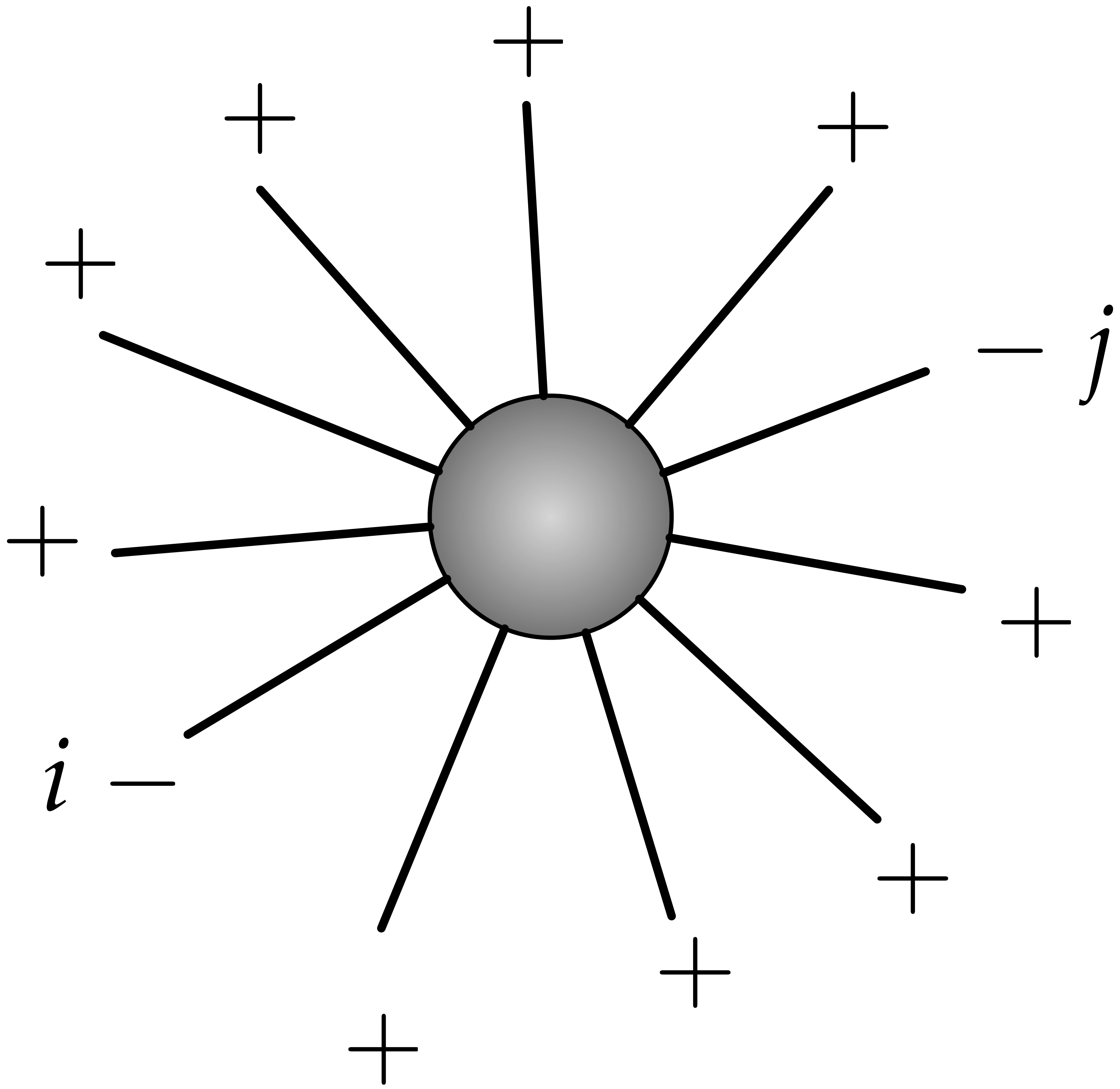
Stephen J. PARKE and T.R. TAYLOR

Fermi National Accelerator Laboratory, P.O. Box 500, Batavia, IL 60510 USA

Received 13 September 1985

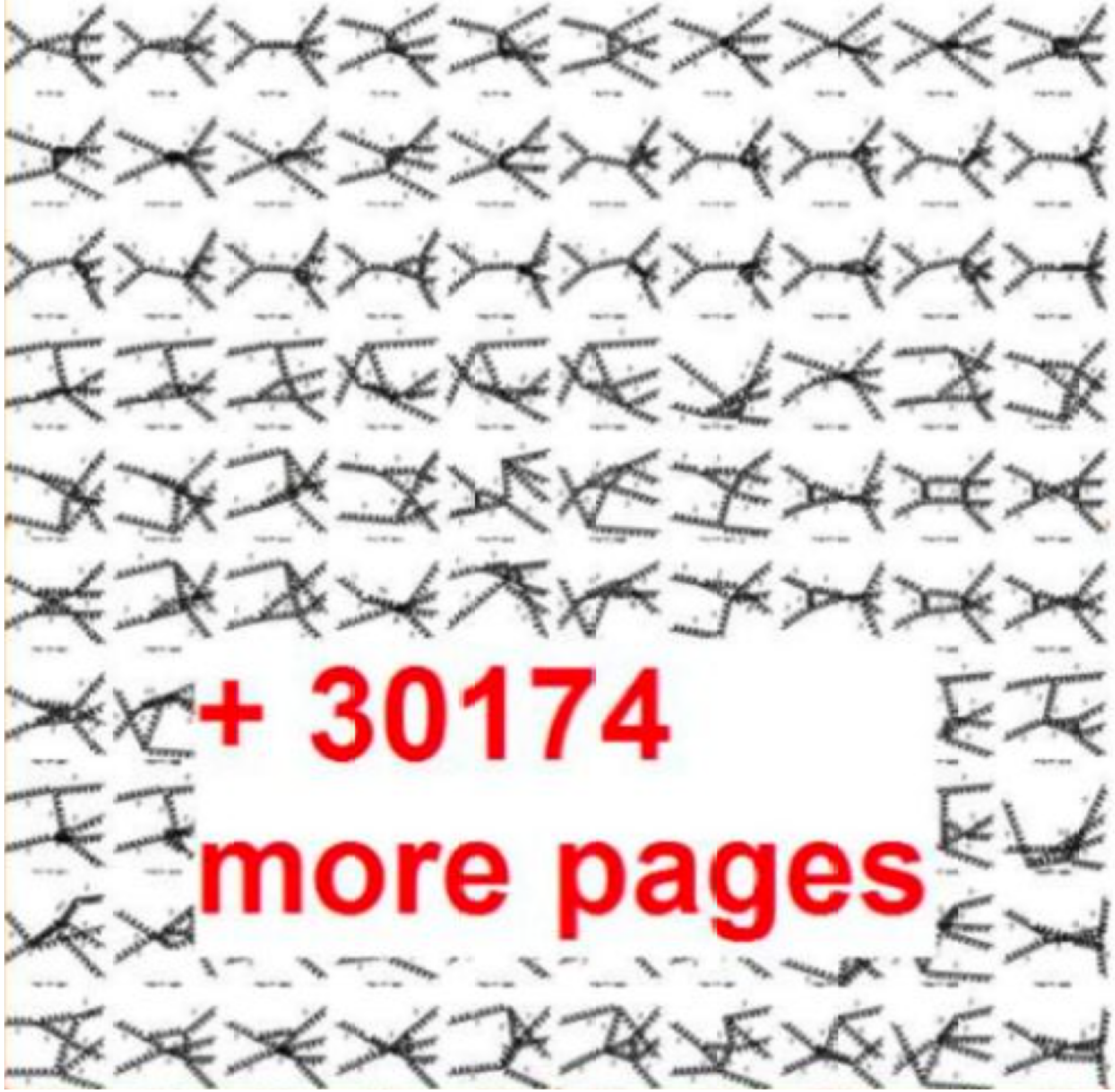
The cross section for two-gluon to four-gluon scattering is given in a form suitable for fast numerical calculations.

Maximally helicity violating (MHV) amplitudes



$$= \frac{\langle ij\rangle^4}{\langle 12\rangle\langle 23\rangle\cdots\langle n1\rangle}\delta^4(P)$$

[Parke-Taylor '86]
[Giele-Berends '88]



[Zvi Bern, KITP colloquium]

One-loop 6-gluon amplitude

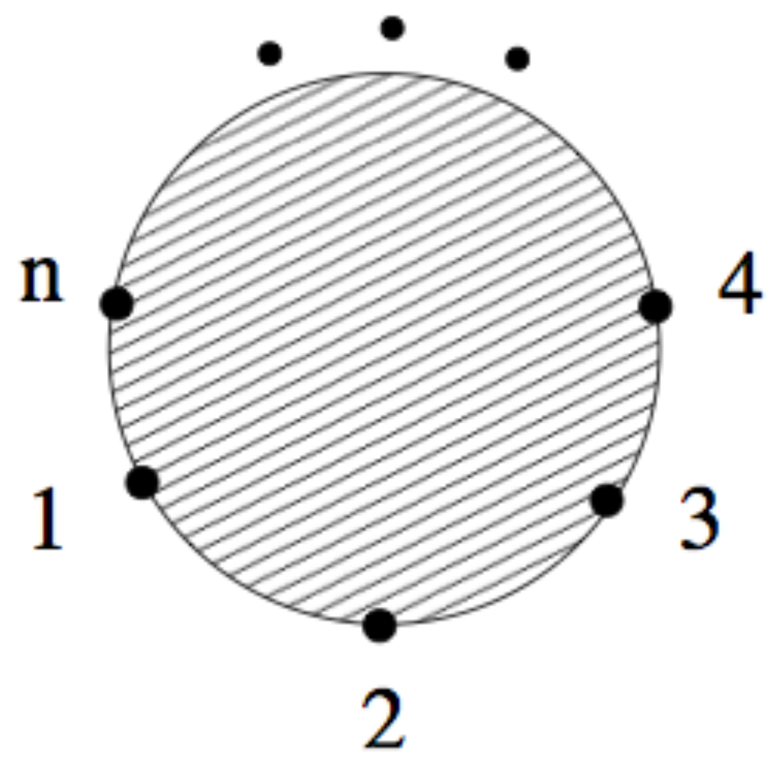
$$= [12345]\mathrm{Li}_2(u) + [12356]\mathrm{Li}_2(v) + [13456]\mathrm{Li}_2(w)$$

(schematically)
in modern notation.

[Marcus Spradlin, talk 2014]

Color decomposition

Color ordered amplitude



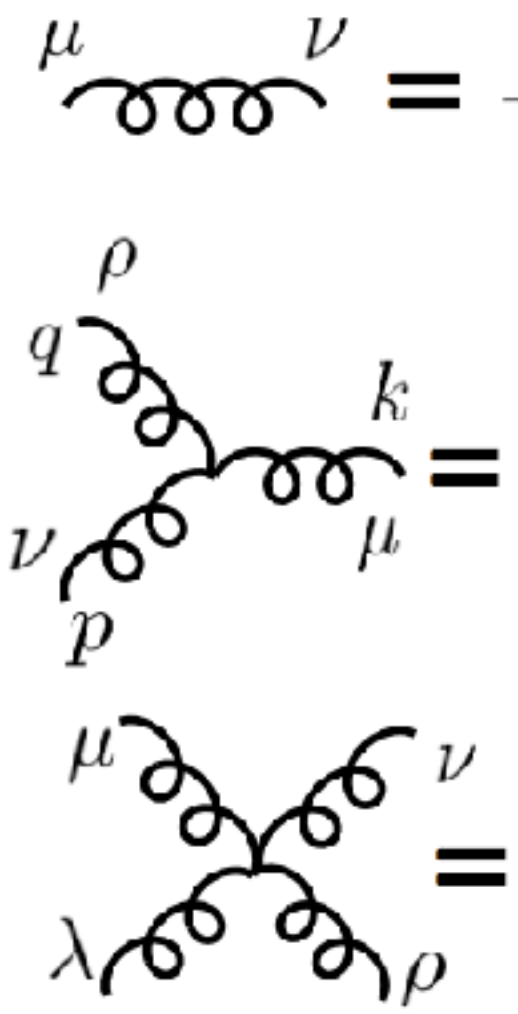
Full amplitude
(permutation symmetry)

"Color-ordered" amplitude
(cyclic symmetry only)

$$\mathcal{M}_n^{a_1,a_2,\ldots,a_n}(p_1,p_2,\ldots,p_n) = \sum_{\sigma\in S_n/\mathbb{Z}_n} \mathrm{Tr}(T^{\sigma(1)}\cdots T^{\sigma(n)})\mathcal{A}_n(\sigma(1),\cdots,\sigma(n))$$

Color ordered Feynman rules

(Less diagrams, simpler Feynman rules)



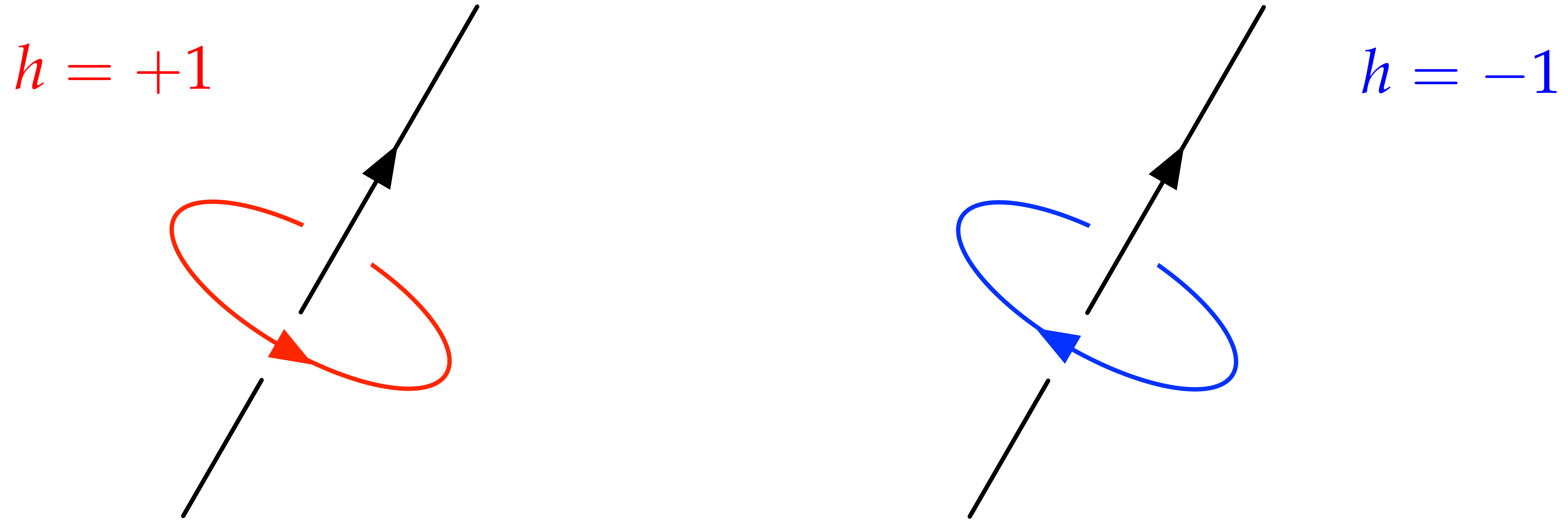
$$= -i\frac{\eta_{\mu\nu}}{p^2}$$

$$= i\sqrt{2}\,(\eta_{\mu\nu}k_\rho + \eta_{\nu\rho}p_\mu + \eta_{\rho\mu}q_\nu)$$

$$= i\eta_{\mu\rho}\eta_{\nu\lambda}$$

Spinor helicity notation

[Berends, Kleiss, Troost, Wu, Xu... 80's]



Massless momentum in bi-spinor notation: $p_{\alpha\dot{\alpha}} = \lambda_\alpha \bar{\lambda}_{\dot{\alpha}}$

Lorentz invariants: $\langle ij \rangle \equiv \epsilon^{\alpha\beta}(\lambda_i)_\alpha(\lambda_j)_\beta\,, \quad [ij] \equiv \epsilon^{\dot{\alpha}\dot{\beta}}(\bar{\lambda}_i)_{\dot{\alpha}}(\bar{\lambda}_j)_{\dot{\beta}}$

Helicity weights: $(\lambda, \bar{\lambda}) \to (t\lambda, t^{-1}\bar{\lambda})\,, \quad \epsilon^+ \to t^{-2}\epsilon^+\,, \quad \epsilon^- \to t^{+2}\epsilon^-$

$$\implies \quad A_n(t_i\lambda_i, t_i^{-1}\bar{\lambda}_i, h_i) = \left(\prod_{i=1}^{n} t_i^{-2h_i}\right) A_n(\lambda_i, \bar{\lambda}_i, h_i)$$

2000's

Tree amplitudes and loop integrands

Loop integrals

Gauge-Gravity-String

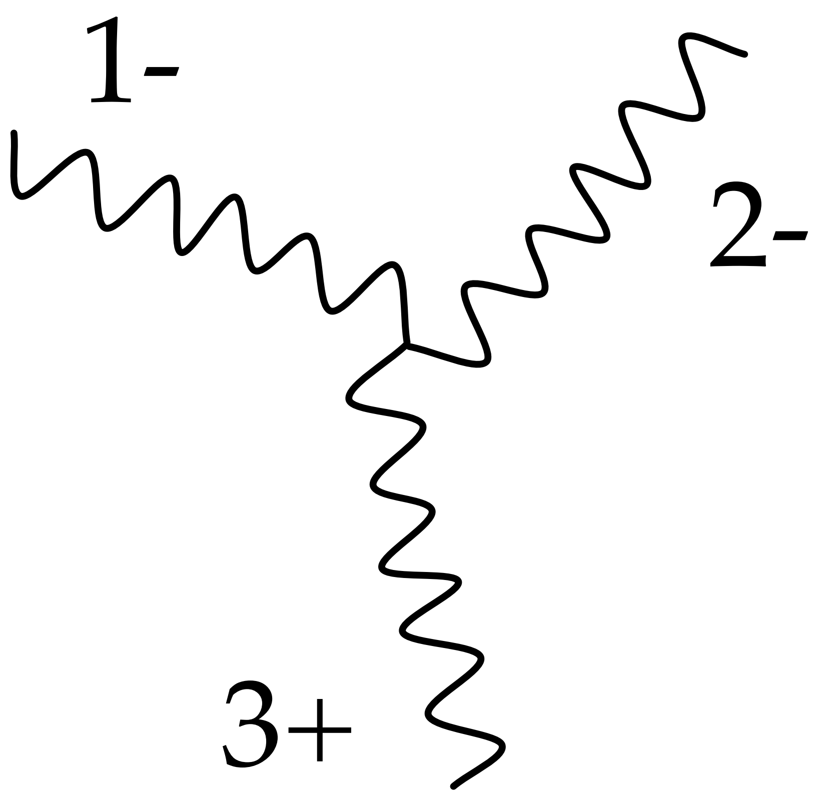
Application to gravitational wave analysis

Tree Amplitudes

&

Loop Integrands

3-gluon amplitude



$$A_3 = \frac{\langle 12\rangle^3}{\langle 23\rangle\langle 31\rangle}\,.$$

$$p_1 + p_2 + p_3 = 0$$

Caution: momentum complexified

A_3 is the unique Lorentz invariant

satisfying the following scaling properties:

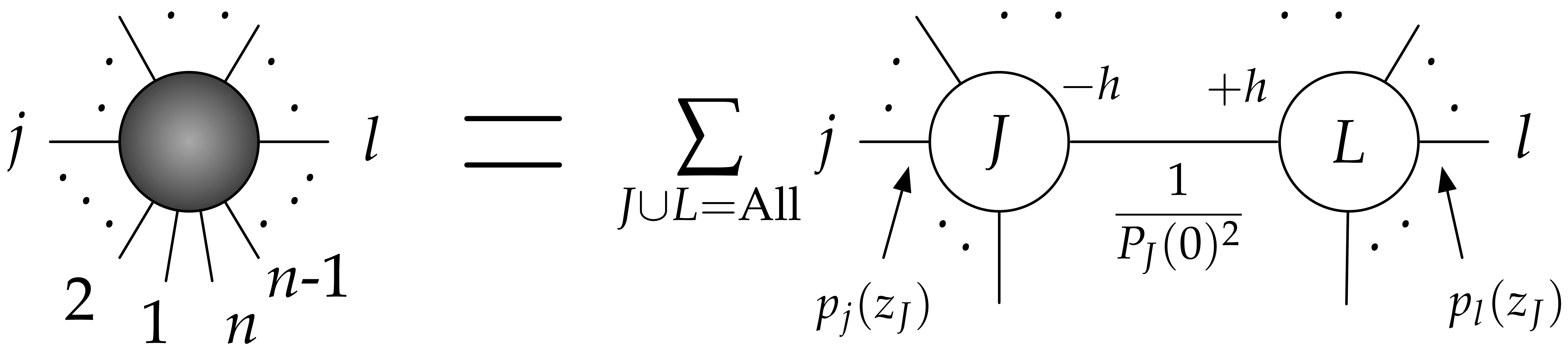
$$A_n(t_i\lambda_i, t_i^{-1}\bar{\lambda}_i; h_i) = \left(\prod_i t_i^{-2h_i}\right) A_n(\lambda_i, \bar{\lambda}_i; h_i)$$

$$A_n(s\lambda_i, s\bar{\lambda}_i; h_i) = s^{-2(n-4)} A_n(\lambda_i, \bar{\lambda}_i; h_i)$$

BCFW recursion

[Britto-Cachazo-Feng(-Witten) 04(05)]

Higher point amplitude from lower ones



Idea: on-shell deformation of momenta

$$\bar{\lambda}_j \to \bar{\lambda}_j - z\bar{\lambda}_l\,, \quad \lambda_l \to \lambda_l + z\lambda_j$$

$(p_j^2 = 0,\ p_l^2 = 0,\ p_1 + \cdots + p_n = 0\ \text{ unaffected})$

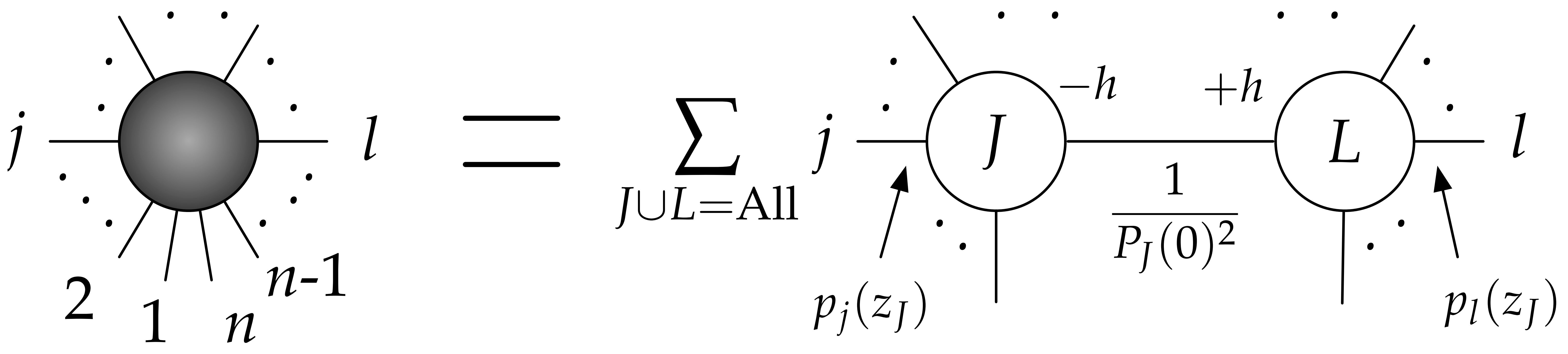
$$A_n = A_n(z=0) = \int \frac{dz}{2\pi i} \frac{A_n(z)}{z}$$

deformed contour,
fall-off at infinity,
residue theorem

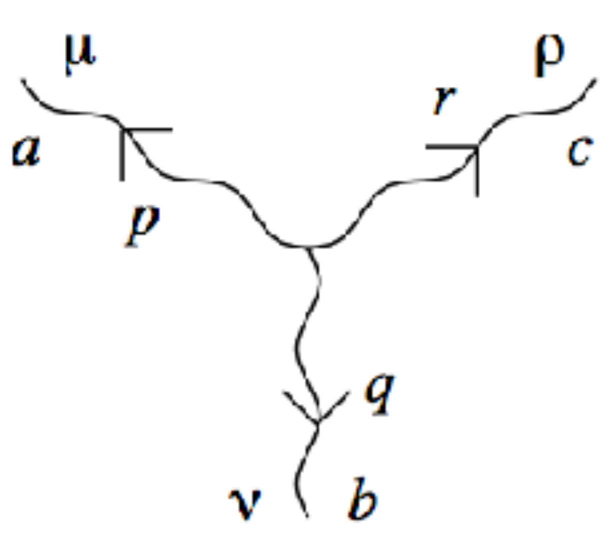
BCFW and gauge symmetry

$$A_n = \sum_{J,\pm} A_J^{\pm}(z_J)\frac{1}{P_J(0)^2}A_L^{\mp}(z_J)$$

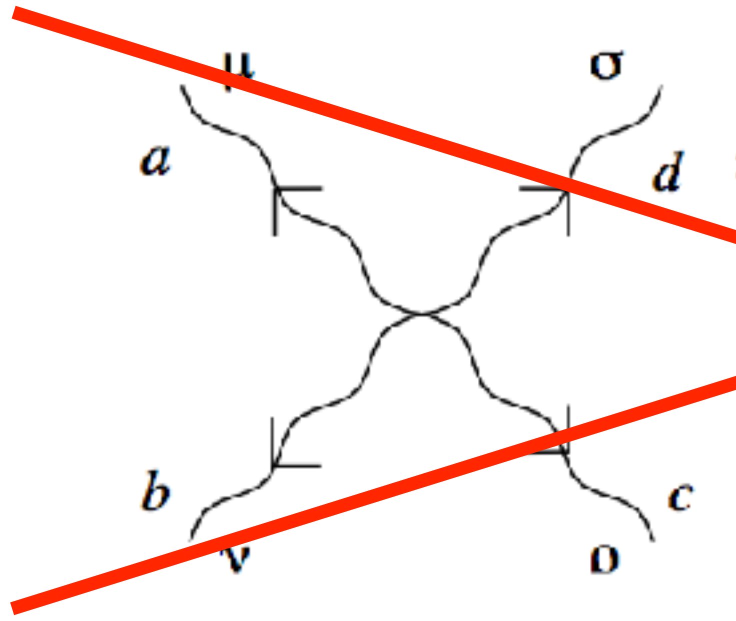
Higher point amplitude from lower ones



No need for the 4-point vertex !



$$= gf^{abc}[(q-r)_\mu g_{\nu\rho} + (r-p)_\nu g_{\rho\mu} + (p-q)_\rho g_{\mu\nu}]$$



$$= -ig^2[f^{abe}f^{cde}(g_{\mu\rho}g_{\nu\sigma} - g_{\mu\sigma}g_{\nu\rho}) + f^{ace}f^{dbe}(g_{\mu\sigma}g_{\rho\nu} - g_{\mu\nu}g_{\rho\sigma}) + f^{ade}f^{bce}(g_{\mu\nu}g_{\sigma\rho} - g_{\mu\rho}g_{\sigma\nu})]$$

Positive Geometry for Amplitudes

Grassmannian integral [Arkani-Hamed,Cachazo,Cheung,Kaplan 09]

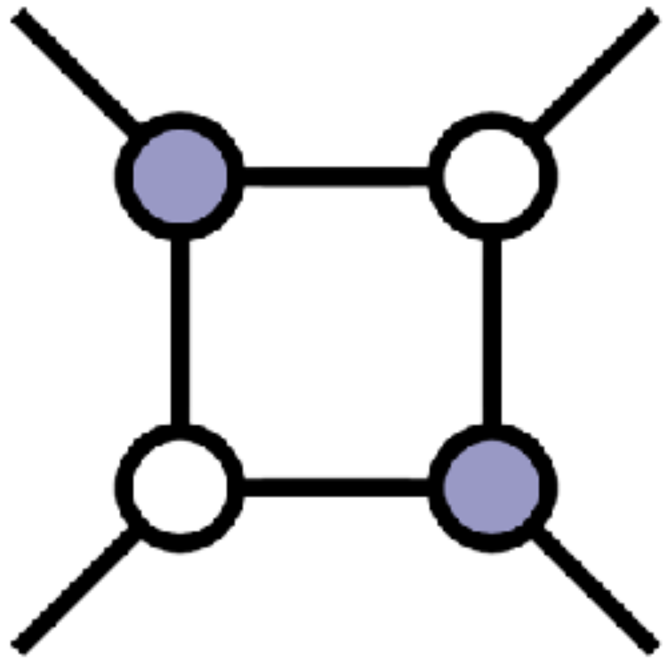
$$\mathcal{A}_{n,k}(\mathcal{W}) = \int \frac{d^{k\times n}C}{\mathrm{vol}\,[GL(k)]}\,\frac{\delta^{4k|4k}(C\cdot\mathcal{W})}{M_1 M_2\cdots M_{n-1}M_n}$$

On-shell diagrams

[Arkani-Hamed,Bourjaily,Cachazo,Goncharov,Postnikov,Trnka 12]

TREE 4-point MHV

=



Amplituhedron [Arkani-Hamed,Trnka 13]

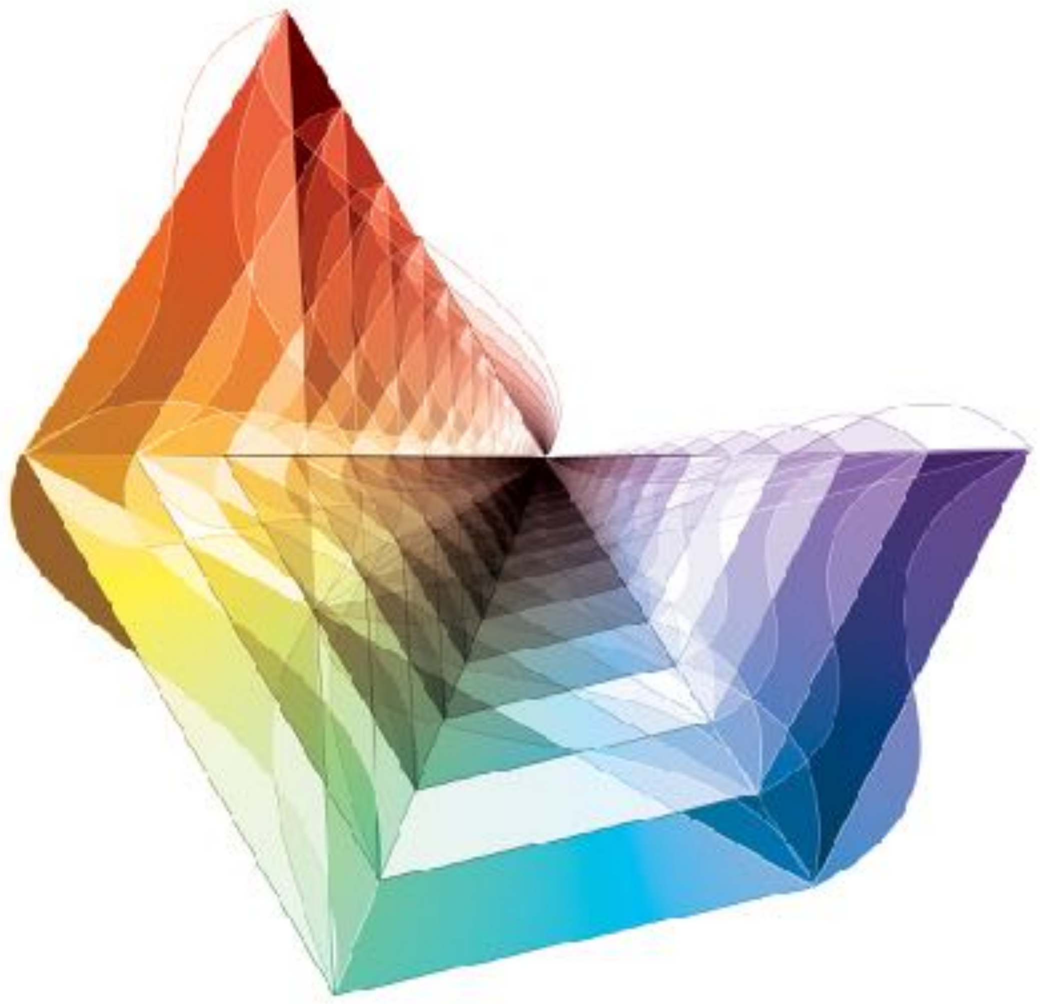


Illustration by Andy Gilmore

BCFW in 3d

[SL 10]
[Gang,Huang,Koh,SL,Lipstein 10]

Momentum conservation in 3d spinor-helicity

$$\sum_{i=1}^{n}(p_i)_{\alpha\beta} = \sum_{i=1}^{n}\lambda_{i\alpha}\lambda_{i\beta} = \left(\lambda_{1\alpha}\cdots\lambda_{n\alpha}\right)\begin{pmatrix}\lambda_{1\beta}\\ \vdots \\ \lambda_{n\beta}\end{pmatrix}$$

3d Grassmannian integral

$$\mathcal{A}_{2k}(\Lambda) = \int \frac{d^{k\times 2k}C}{\mathrm{vol[GL(k)]}}\frac{\delta(C_{mi}C_{ni})\prod_{m=1}^{k}\delta^{2|3}(C_{mi}\Lambda_i)}{M_1(C)M_2(C)\cdots M_k(C)}$$

Gauge-Gravity-String

KK identities

[Kleiss-Kuijf 89]

Color ordered amplitude

$$\mathcal{A}_n(p_i, h_i, a_i) = \sum_{\sigma \in S_n/\mathbb{Z}_n} \mathrm{Tr}(T^{a_{\sigma(1)}} \cdots T^{a_{\sigma(n)}}) \mathcal{A}_n(\sigma(1^{h_1}), \cdots, \sigma(n^{h_n}))$$

$$f^{abc} = \mathrm{Tr}(T^a T^b T^c) - \mathrm{Tr}(T^b T^a T^c), \quad f^{abe} f_e{}^{cd} + f^{bce} f_e{}^{ad} + f^{cae} f_e{}^{bd} = 0$$

#(independent amplitudes) : $(n-1)! \to (n-2)!$

Color-Kinematics duality

[Bern,Carrasco,Johansson 08]

$$\mathcal{A}_4 = \frac{n_s c_s}{s} + \frac{n_t c_t}{t} + \frac{n_u c_u}{u}$$

$$c_s + c_t + c_u = 0$$ color (Jacobi)

$$n_s + n_t + n_u = 0$$ kinematics (p-cons)

$$\left(\mathcal{A}_n(\text{gauge}) = \sum_I \frac{c_I n_I}{D_I} \implies \mathcal{M}_n(\text{gravity}) = \sum_I \frac{n_I \tilde{n}_I}{D_I}\right)$$ BCJ doubling

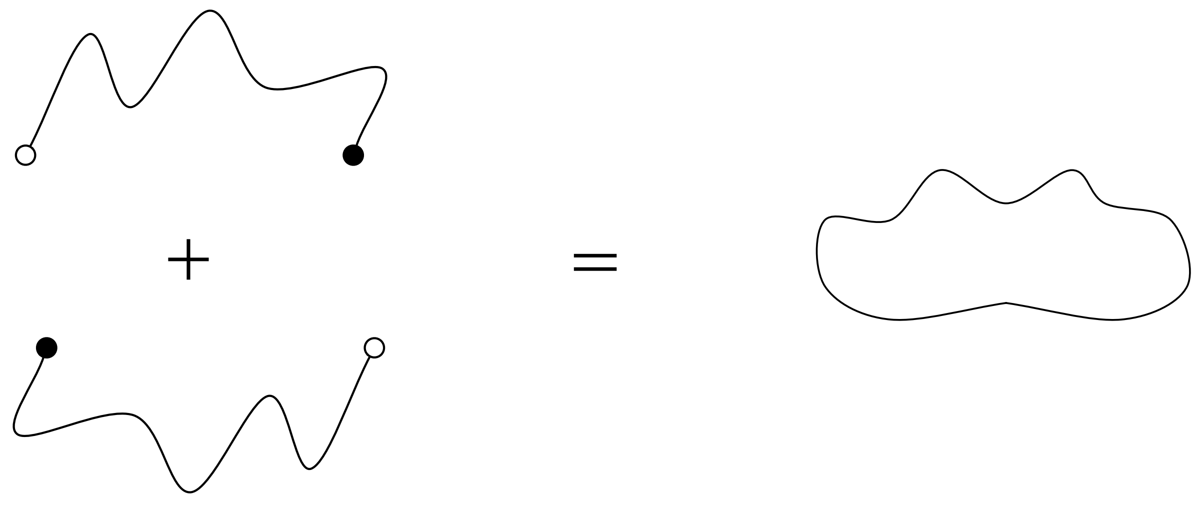
$$\#(\text{independent amplitudes}): \quad (n-2)! \to (n-3)!$$

KLT doubling

[Kawai-Lewellen-Tye 86]

$$[A_n(\text{gauge})]^2 = A_n(\text{gravity})$$

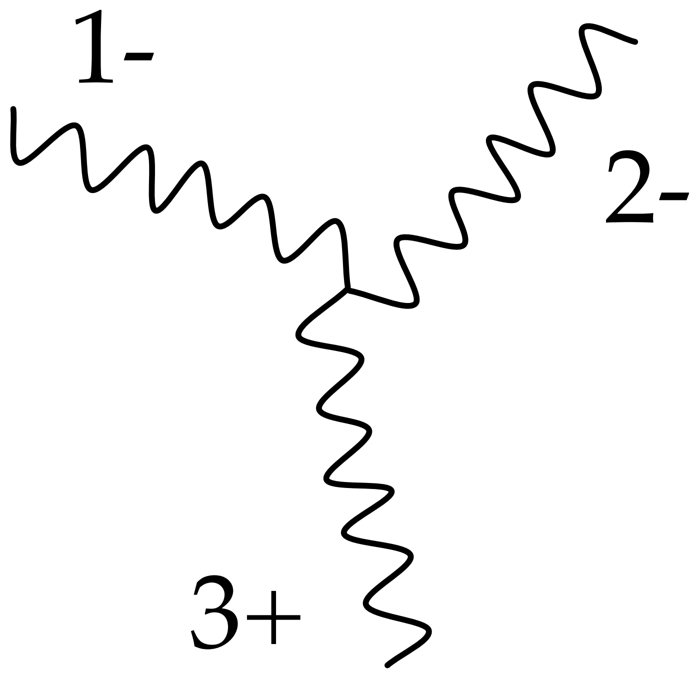
"(Open)^2 = (Closed)"

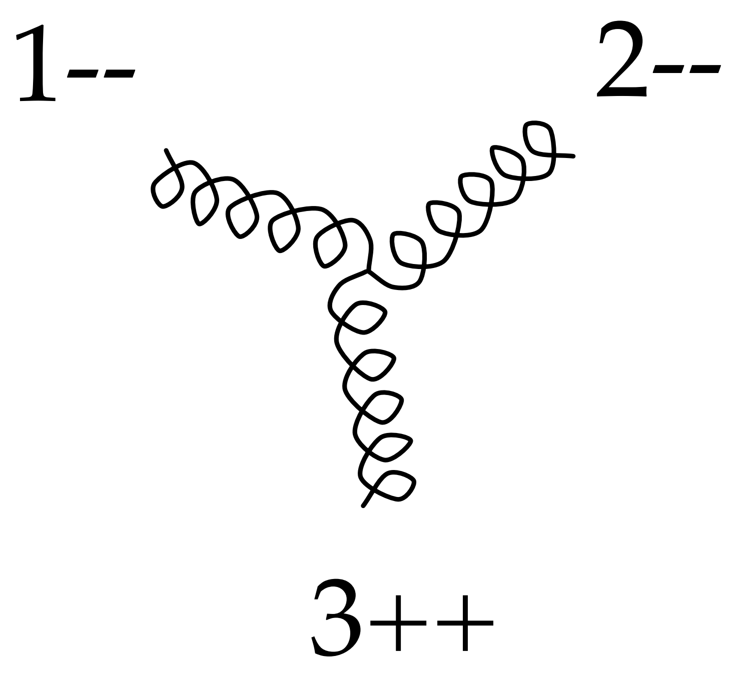


(massless limit)

- Ex) 3-graviton amplitude

$$\mathcal{A}_3(\text{gauge}) = \frac{\langle 12\rangle^3}{\langle 13\rangle\langle 23\rangle} \qquad \mathcal{A}_3(\text{gravity}) = \left(\frac{\langle 12\rangle^3}{\langle 13\rangle\langle 23\rangle}\right)^2$$





Scattering equation

[Cachazo,He,Yuan 13]

$$\sum_{b\neq a} \frac{p_a \cdot p_b}{z_a - z_b} = 0 \quad (z_a \in \mathbb{CP}^1)$$

$$\mathcal{M}_n^{(\mathbf{s})} = \int \frac{d^n\sigma}{\text{vol}\,\text{SL}(2,\mathbb{C})} \prod_a {}'\delta\Big(\sum_{b\neq a} \frac{s_{ab}}{\sigma_a - \sigma_b}\Big) \left(\frac{\text{Tr}(T^{\mathsf{a}_1} T^{\mathsf{a}_2} \cdots T^{\mathsf{a}_n})}{(\sigma_1 - \sigma_2)\cdots(\sigma_n - \sigma_1)} + \ldots \right)^{2-\mathbf{s}} (\text{Pf}'\Psi)^{\mathbf{s}}$$

Valid for massless spin 0, 1, 2 particles in all dimensions

Generalizes Witten's twistor string theory

Same equation appears in the high energy string scattering [Gross,Mende 87]

Loop Integrals

Taming the loop integrals

Traditional methods

Reduction to scalar integrals

Relation between similar integrals

... basis of "master" integrals

integration by parts relations
differential equations
difference equations

Modern technology

symbols, Hopf algebra, coaction,
zeta values, single-valued projection, ...

log, poly-log (harmonic, multiple, elliptic ...)

Taming the loop integrals

Poly-logarithm :

Nested integral of

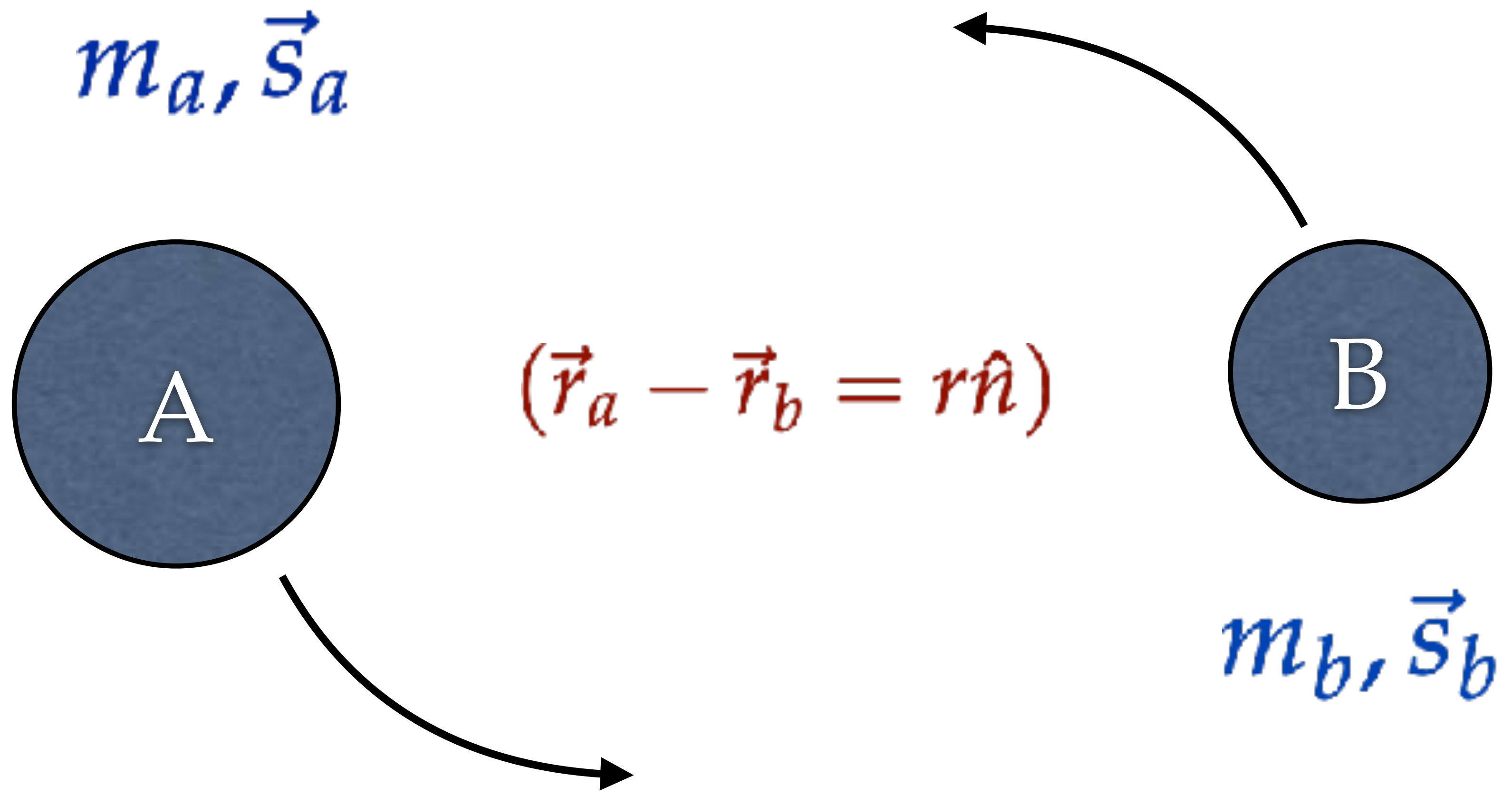
$$d\log f_1 \wedge d\log f_2 \wedge \cdots \wedge d\log f_n$$
$$= \frac{df_1}{f_1} \wedge \frac{df_2}{f_2} \wedge \cdots \wedge \frac{df_n}{f_n}$$

Example : scalar box integral

$$\frac{d^4\ell \;\; (p_1+p_2)^2(p_1+p_3)^2}{\ell^2(\ell+p_1)^2(\ell+p_1+p_2)^2(\ell-p_4)^2}$$
$$= d\log\left(\frac{\ell^2}{(\ell-\ell^*)^2}\right) d\log\left(\frac{(\ell+p_1)^2}{(\ell-\ell^*)^2}\right) d\log\left(\frac{(\ell+p_1+p_2)^2}{(\ell-\ell^*)^2}\right) d\log\left(\frac{(\ell-p_4)^2}{(\ell-\ell^*)^2}\right)$$

Application to Gravitational Wave

Binary system with spin



Newton : $$H = \frac{1}{2m_a}\vec{p}_a^{\,2} + \frac{1}{2m_b}\vec{p}_b^{\,2} - \frac{G m_a m_b}{r}\,.$$

Einstein : $$\mathcal{S} = \frac{1}{16\pi G}\int d^4x\sqrt{-g}R \,+\, \mathcal{S}_{\text{matter}}\,.$$

Newton ~ leading approximation to Einstein

PN vs PM

(1686)	(1938)	(1974)	(2000)
0PN	1PN	2PN	3PN

$$\left(\frac{Gm}{r}\right)\left(1 + v^2 + v^4 + v^6 + v^8 + \cdots\right)$$ 1PM

$$\left(\frac{Gm}{r}\right)^2\left(1 + v^2 + v^4 + v^6 + \cdots\right)$$ 2PM

$$\left(\frac{Gm}{r}\right)^3\left(1 + v^2 + v^4 + \cdots\right)$$ 3PM

$$\left(\frac{Gm}{r}\right)^4\left(1 + v^2 + \cdots\right)$$ 4PM

$$\vdots$$

QFT-inspired methods pioneered by

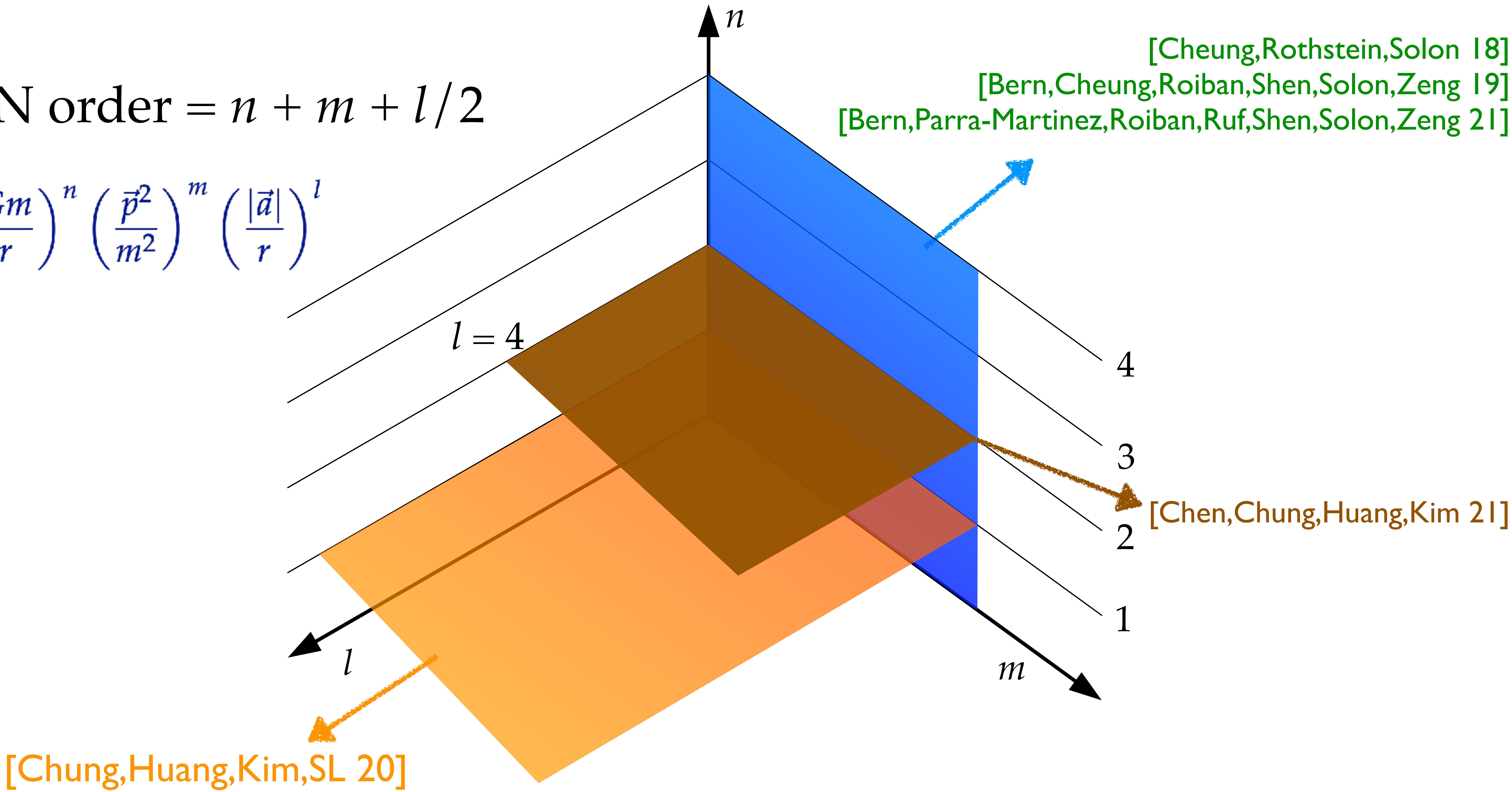
[Donoghue 96]
[Goldberger, Rothstein 04]
[Hostein,Ross 08]
[Neill,Rothstein 13]

became main-stream around 2018

Amplitude approach : the frontier

PN order $= n + m + l/2$

$$\left(\frac{Gm}{r}\right)^n \left(\frac{\vec{p}^2}{m^2}\right)^m \left(\frac{|\vec{a}|}{r}\right)^l$$



Outlook

Quantum Field Theory

We have learned a lot about QFT over a century.

New discoveries are still being made, even in perturbation theory.

Gauge theory, gravity, string theory are closely related.

More experts are needed in the Korean community.

Now may be a good time to update "how to teach QFT".